SEMIPARAMETRIC ESTIMATION OF LONG-TERM TREATMENT EFFECTS

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This paper studies the estimation of long-term treatment effects through the combination of short-term experimental and long-term observational datasets. In particular, we consider settings in which only short-term outcomes are observed in an experimental sample with exogenously assigned treatment, both short-term and long-term outcomes are observed in an observational sample where treatment assignment may be confounded, and the researcher is willing to assume that the causal relationships between treatment assignment and the short-term and long-term outcomes share the same unobserved confounding variables in the observational sample. We derive the efficient influence function for the average causal effect of treatment on long-term outcomes in each of the models that we consider and characterize the corresponding asymptotic semiparametric efficiency bounds.

1. Introduction. Empirical researchers often aim to estimate the long-term effects of a policy or intervention. Randomized experimentation is a powerful approach to this problem, providing a simple and interpretable solution to selection bias (Fisher, 1925, 1935; Duflo, Glennerster and Kremer, 2007; Athey and Imbens, 2017). However, the long-term outcomes of an experimental evaluation are necessarily observed only after a long delay. As a result, there is limited evidence from randomized evaluations on the long-term effects of economic and social policies. Bouguen, Huang, Kremer and Miguel (2019) provide a systematic review of existing randomized control trials in development economics and report that only a small proportion evaluate long-term effects. Similarly, in a review of experimental evaluations of the effects of early childhood educational interventions, Tanner, Candland and Odden (2015) are able to identify only one randomized evaluation that reports long-term employment and labor market effects (Gertler et al., 2014).

In this paper, we consider the problem of estimating long-term treatment effects by combining short-term experimental and long-term observational data. In particular, we consider two closely related settings originally proposed by Athey, Chetty and Imbens (2020a) and Athey, Chetty, Imbens and Kang (2020b). In both settings, a researcher is interested in the long-term causal effect of a binary treatment on a scalar long-term outcome. The researcher observes two samples of data—one sample contains measurements of the short-term outcomes of a randomized evaluation of a treatment of interest and the other sample contains observational measurements of the joint distribution of short-term and long-term outcomes. The researcher is willing to assume that the short-term and long-term outcomes share the same unobserved confounding variables in the observational sample. Athey et al. (2020a) consider the case where treatment is observed in the observational sample, whereas Athey et al. (2020b) consider the case where treatment is not observed in the observational sample.

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This setting is central to many contexts in applied microeconomics. Leading examples are given by evaluations of the long-term effects of early childhood interventions and job training programs; see Tanner et al. (2015) and Heckman, LaLonde and Smith (1999) for reviews, respectively. More specifically, the estimators developed in Athey et al. (2020b) have already been applied to estimation of the long-term effects of free tuition on college completion (Dynarski, Libassi, Michelmore and Owen, 2019) and to estimation of the long-term effects of an agricultural monitoring technology on farmer revenue in rural Paraguay (Dal Bó, Finan, Li and Schechter, 2021). Additionally, measurement of long-term treatment effects has been highlighted as a “top challenge” for industrial practitioners who aim to evaluate the impacts of changes to software products and services with online controlled experiments or A/B tests (Gupta et al., 2019). Our objective is to provide a formal analysis of the semiparametric properties of these models, and thereby a more precise characterization of the structure of the uncertainty in the problem.

We begin in Section 2 by formulating the models that we consider and stating a set of assumptions sufficient for the identification of the estimand of interest—the long-term treatment effect. Settings closely related to our own include Rosenman, Owen, Baiocchi and Banack (2018), Rosenman, Basse, Owen and Baiocchi (2020), and Kallus and Mao (2020). In Rosenman et al. (2018), treatment assignment is unconfounded in both samples. In Rosenman et al. (2020), the outcome of interest is observed in both samples and the authors consider shrinkage estimators. In Kallus and Mao (2020), treatment is unconfounded in both samples considered jointly, but not necessarily in either sample considered individually. Yang, Eckles, Dhillon and Aral (2020) study the design of optimal long-term policies in a setting related to Athey et al. (2020b).

In Section 3, we derive the efficient influence function for the long-term treatment effect when treatment is and is not observed in the long-term observational sample. In both cases, we follow the approach presented in Section 3.4 of Bickel, Klaassen, Ritov and Wellner (1993) by characterizing the tangent space of each model at the data generating distribution, demonstrating that conjectured influence functions are elements of these spaces, and verifying that the conjectured functions satisfy sets of necessary conditions that characterize influence functions in each case. With these results, we compute the semiparametric efficiency bounds for the long-term treatment effect; that is, in both cases, we derive the minimum asymptotic variance for any regular and asymptotically linear estimator of the long-term treatment effect (van der Vaart, 1989). Our characterization of the efficient influence functions and semiparametric efficiency bounds for these models are novel.

Moreover, we demonstrate that these influence functions are the only influence functions for the long-term treatment effect in these models, under the identification assumptions that we make. This uniqueness implies that all regular and asymptotically linear estimates of long-term treatment effects in these models are first-order equivalent and attain their respective semiparametric efficiency bounds (Chen and Santos, 2018). The uniqueness also implies that the identification assumptions maintained are in some sense minimal.

This paper contributes to the literature on missing data models (Little and Rubin, 2019; Ridder and Moffitt, 2007; Hotz, Imbens and Mortimer, 2005), and more specifically to the literature on semiparametric efficiency in missing data models (Chen, Hong and Tarozzi, 2018).

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1See Bickel et al. (1993), Chapter 25 of Van der Vaart (2000), or Newey (1994) for presentations of the theory of semiparametric asymptotic efficiency.

2Our result corrects the statements of the influence function and semiparametric efficiency bounds given in Theorem 1 and Theorem 3 of the February, 2020 draft of Athey et al. (2020b).

On June 15th, 2021, Nathan Kallus, in collaboration with Xiaojie Mao, independently presented the efficient influence function for the model that we consider in a discussion of Athey et al. (2020a) in the Online Causal Inference Seminar.
2008; Graham, 2011; Muris, 2020). The models that we consider are related to the literatures on statistical surrogacy (Prentice, 1989; Begg and Leung, 2000; Weir and Walley, 2006; Xu and Zeger, 2001; Freedman, Graubard and Schatzkin, 1992) and mediation analysis (Baron and Kenny, 1986; van der Laan and Petersen, 2004; Imai, Keele and Tingley, 2010).

Section 4 concludes. Proofs for all Theorems stated in the main text are provided in Appendix A. Appendix B gives supplementary results that will be introduced at appropriate points throughout the paper.

2. Problem Formulation. In this section, we formulate the problem of interest—estimation of the average effect of a treatment of interest on a long-term outcome. We begin by defining the notation maintained throughout the paper. We then define the structure of the data available in our models of interest and a specify a set of assumptions sufficient for the identification of the long-term treatment effect in different contexts.

2.1. Notation. Let $\nu$ be a $\sigma$-finite probability measure on the measurable space $(\Omega, \mathcal{F})$, and let $\mathcal{M}_\nu$ be the set of all probability measures on $(\Omega, \mathcal{F})$ that are absolutely continuous with respect to $\nu$. For an arbitrary random variable $D$ defined on $(\Omega, \mathcal{F}, \nu)$, we let $\mathbb{E}_\nu[D]$ denote its expected value. We let $p(d \mid E)$ denote the density of the arbitrary random variable $D$ at $d$ under $\nu$ with respect to $E \in \Omega$ and let

$$\ell(d \mid E) = \log p(d \mid E).$$

denote the log-likelihood at $Q$ of $d$ conditional on $E$. The density notation leaves the law of $Q$ of the random variable $D$ implicit, but causes no ambiguity.

For a general functional $\varphi$ on $\mathcal{M}_\nu$ and an arbitrary random variable $D$ with distribution $Q$ in $\mathcal{M}_\nu$, we write $\varphi(Q) = \varphi(D)$ interchangeably and denote the pathwise derivative of $\varphi$ at $\zeta$ on some regular parametric submodel $\mathcal{Q} = \{Q_\zeta : \zeta \in [-1, 1]\}$ of $\mathcal{M}_\nu$ by $\varphi'(D; \zeta)$. Unless otherwise noted, we let $\varphi'(D) = \varphi'(D; 0)$ denote the pathwise derivative evaluated at $\zeta = 0$.

2.2. Data. Consider an independent and identically distributed sample of potential outcomes

$$\{A_i\}_{i=1}^n = \{(Y_i(0), Y_i(1), S_i(0), S_i(1), W_i, G_i, X_i)\}_{i=1}^n$$
drawn according to a distribution $P_\nu \in \mathcal{M}_\nu$ with support $A = (\mathbb{Y}^2, S^d, W, G, X)$, where $\mathbb{Y} = \mathbb{R}$, $S = \mathbb{R}^d$, $W = G = \{0, 1\}$, and $X = \mathbb{R}^P$. In this formulation, $X_i$ is a $p$-vector of covariates, $G_i$ is a binary indicator denoting whether the observation was acquired in an observational sample as opposed to an experimental sample, $W_i$ is a binary treatment indicator, $S_i$ is a $d$-vector of short-term auxiliary outcomes, and $Y_i$ is a long-term outcome of interest.

We let $Y_i(w)$ and $Y_i(w)$ denote the short-term and long-term outcomes that would have been observed under treatment $w$; that is, observed outcomes $S_i$ and $Y_i$ are given by

$$S_i = W_iS_i(1) + (1 - W_i)S_i(0) \quad \text{and} \quad Y_i = W_iY_i(1) + (1 - W_i)Y_i(0),$$

respectively. The short-term outcome is observed in both the observational and experimental samples. Treatment is always observed in the experimental sample. The long-term outcome is observed only in the observational sample. We consider both cases in which the treatment is and is not observed in the observational sample. In short, we denote the observed sample by

$$\{B_i\}_{i=1}^n = \{(Y_i, S_i, W_i, G_i, X_i)\}_{i=1}^n,$$

assume that it is drawn from the distribution $P \in \mathcal{M}_\nu$, which is consistent with the distribution $P_\nu$ of the potential outcomes, and consider several alternative sets of assumptions on $P$. 

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2.3. Estimands and Identification. In the main text, we consider estimation of the long-term average treatment effect for individuals in the observational sample, given by

\[ \tau_1 = \mathbb{E}_{P^*} [Y_i(1) - Y_i(0) \mid G_i = 1] . \]  

(2.1)

In Appendix B, we give results analogous to those presented in the main text for the long-term treatment effect for units in the experimental sample

\[ \tau_0 = \mathbb{E}_{P^*} [Y_i(1) - Y_i(0) \mid G_i = 0] , \]  

(2.2)

which may be of interest in some contexts. \(^4\) Athey \textit{et al.} (2020a) and Athey \textit{et al.} (2020b) consider the estimation of \(\tau_1\) and \(\tau_0\) when treatment is and is not observed in the observational sample, respectively. For the most part, we operate under the identifying assumptions introduced in these papers, but impose an additional restriction to ensure that \(\tau_1\) is identified when treatment is not observed in the observational sample.

First, we assume that treatment assignment is unconfounded in the experimental sample.

**Assumption 2.1** (Experimental Unconfounded Treatment). In the experimental sample, treatment is independent of both short-term and long-term outcomes conditional on pretreatment covariates:

\[ W_i \perp (Y_i(0), S_i(0), Y_i(1), S_i(1)) \mid X_i, G_i = 0 . \]

This assumption is satisfied if the experimental sample contains the outcomes of a randomized experiment, where the probability of being assigned to treatment may depend on covariates.

Next, we impose a restriction that allows for comparison between the experimental and observational samples. The form of this assumption differs somewhat based on whether we do or do not observe treatment in the observational sample. If treatment is observed in the observational sample, we assume that the distribution of the potential outcomes is independent of whether the data belong to the experimental or observational samples.

**Assumption 2.2** (Experimental Conditional External Validity). The experimental sample has conditional external validity, in the sense that

\[ G_i \perp (Y_i(1), Y_i(0), S_i(1), S_i(0)) \mid X_i . \]

In other words, this assumption imposes the restriction that there are no systematic differences in average effects between the experimental and observational data sets, conditional on covariates, that are not due to unobserved confounding in the observational sample.

Similarly, if treatment is not observed in the observational sample, we additionally assume that long-term observed outcomes are independent of whether the unit is assigned to the experimental or observational sample conditional on the short-term observed outcomes and covariates.

**Assumption 2.3** (Long-term Outcome Comparability). The distributions of the long-term outcome are comparable between the experimental and observational samples conditional on the short-term outcome and covariates, in the sense that

\[ G_i \perp Y_i \mid X_i, S_i . \]

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\(^3\) Note that we use \(\mathbb{E}_{P^*}, P^*,\) or \(p^*\) (where \(p^*\) is the density of \(P^*\) with respect to \(\nu\)) to emphasize that a certain expectation, probability, or density is with respect to objects defined in the complete data (e.g. potential outcomes, which are sometimes not observed).

\(^4\) Alternatively, we could consider estimation of the unconditional long-term average treatment effect \(\tau = \mathbb{E}_{P^*} [Y_i(1) - Y_i(0)]\). However, the practical interpretation of this parameter is somewhat nebulous, as it is unclear why it would be desirable to weight the two samples in the definition of the parameter according to their sizes. By contrast, \(\tau_1\) and \(\tau_0\) have clear interpretations as average treatment effects on observed and experimentally collected samples.
Observe that Assumption 2.2 is not strictly stronger than Assumption 2.3, as being in the experimental or observational sample is not independent of treatment assignment.

Finally, we must impose a restriction that allows the short-term outcomes to serve as a valid proxy for the long-term outcome. Again, this restriction takes a different form depending on whether we observe treatment in the observational sample. If we observe treatment in the observational sample, we assume that the effects of the treatment on the short-term and long-term outcomes have the same set of unobserved confounding variables.

**Assumption 2.4 (Observational Latent Unconfounded Treatment).** In the observational sample, treatment is independent of the long-term outcomes conditional on the short-term outcomes and pretreatment covariates, in the sense that, for \( w \in \{0, 1\} \),

\[
W_i \perp Y_i(w) \mid S_i(w), X_i, G_i = 1.
\]

Mechanically, this assumption amounts to adding the short-term potential outcomes to the unconfounded treatment condition in the experimental sample. If treatment is not observed in the observational sample, an alternative “surrogacy” assumption in the spirit of Prentice (1989) is necessary.

**Assumption 2.5 (Experimental Surrogacy).** In the experimental sample,

\[
W_i \perp Y_i \mid S_i, X_i, G_i = 0.
\]

In effect, this condition imposes the restriction that the short-term outcomes fully capture the causal effect of the treatment on the long-term outcomes.\(^5\)

Under the preceding assumptions the estimand of interest \( \tau_1 \) is point-identified if we observe or do not observe the treatment in the observational sample. Parts of the following theorem are stated in Theorem 1 of Athey et al. (2020a) and Theorem 1 of Athey et al. (2020b). We state and prove the result for completeness.

**Theorem 2.1.** The estimand of interest \( \tau_1 \), defined in Equation (2.1), is point-identified under the following conditions:

1. (Athey et al., 2020a) Under Assumptions 2.1, 2.2, and 2.4 and if treatment is observed in the observational sample, then \( \tau_1 \) is point identified.
2. (Athey et al., 2020b) Under Assumptions 2.1 to 2.3 and 2.5 and if treatment is not observed in the observational sample, then \( \tau_1 \) is point identified.

**Remark 2.1.** Figure 1 displays a causal Directed Acyclic Graph (DAG) consistent with the maintained assumptions on the observational data; see Pearl (1995) for further discussion of the application of graphical models of this form to problems in causal inference.

**3. Semiparametric Efficiency.** In this section, we derive efficient influence functions and the corresponding semiparametric efficiency bounds for the estimation of the long-term treatment effect on units in the observational sample \( \tau_1 \) in the models formulated in Section 2. We treat the cases that treatment is and is not observed separately. In Appendix B, we state analogous results in both cases for the long-term treatment effect on units in the experimental sample \( \tau_0 \).

\(^5\)In practice, Assumptions 2.4 and 2.5 are quite strong as there may be many determinants of long-term outcomes that may impact treatment but not play a role in short-term outcomes. Athey et al. (2020b) provide a procedure for bounding the bias induced in estimates of \( \tau_0 \) if Assumption 2.5 is violated under a set of functional form assumptions. We see the extension of these methods as a useful direction for further research.
fig 1. Causal DAGs Consistent with Assumptions on Observational Data

Panel A: Treatment Observed

Panel B: Treatment Not Observed

Notes: Figure 1 displays causal DAGs consistent with the maintained assumptions on the observational sample (G = 1). Panels A and B give the assumptions for the case in which treatment is observed and is not observed in the observational sample, respectively. Grey arrows denote the existence of causal effect of the tail variable on the head variable. Dark red arrows denote that a causal effect of the tail variable on the head variable is ruled out by Assumption 2.4 and Assumption 2.5 in Panels A and B, respectively. Dashed bidirectional arrows represent existence of some unobserved common causal variable U, where we have a fork ← U →.

3.1. Treatment is Observed in the Observational Sample. We begin by considering the case that treatment is observed and focusing attention on the parameter \( \theta_{11} \), defined more generally by \( \theta_{w,g} = \mathbb{E}[Y_i(w) | G_i = g] \), where we note that \( \tau_1 = \theta_{11} - \theta_{01} \).

Towards this end, we introduce additional notation, collected in Table 1. Let

\[
\mu_w(s, x) = \mathbb{E}_P[Y | S = s, W = w, G = 1, X = x]
\]

denote expectation of the long-term outcome in the observational sample conditional on the short-term outcome and covariates and let

\[
\mu_w(x) = \mathbb{E}_P[\mu_w(S, x) | W = w, G = 0, X = x]
\]

denote the expectation of \( \mu_w(S, x) \) in the experimental sample conditional on covariates. Observe that by Assumptions 2.1, 2.2, and 2.4, we can express \( \mu_w(s, x) \) and \( \mu_w(x) \) as functionals of the complete data distribution by

\[
\mu_w(s, x) = \mathbb{E}_{P_x}[Y(w) | S(w) = s, X = x] \quad \text{and} \quad \mu_w(x) = \mathbb{E}_{P_x}[Y(w) | X = x],
\]

respectively. Additionally, let the latent propensity score be

\[
q_w(s, x) = P_x(W = w | S(w) = s, G = 1, X = x),
\]

which denotes the treatment probability in the observational sample conditional on the short-term outcome and covariates. The latent propensity score \( q_w(s, x) \) conditions on an unobserved random variable \( S(w) \); however, observe that by Assumption 2.1 and Assumption 2.2, we may write \( q_w(s, x) \) in terms of observables

\[
q_w(s, x) = \frac{p_x(s(w) | W = w, G = 1, X = x)}{p_x(s(w) | W = w, G = 0, X = x)} P_x(W = w | X = x, G = 1)
\]

\[
= \frac{P(G = 1 | S = s, W = w, X = x)}{P(G = 0 | S = s, W = w, X = x)} P(W = w | X = x, G = 1),
\]

where \( p_x \) denotes the density of \( P_x \) with respect to \( \nu \). Finally, let

\[
\gamma(x) = P_x(G = 1 | X = x), \quad e(x) = P_x(W = 1 | G = 0, X = x), \quad \text{and} \quad \pi = P_x(G = 1)
\]
We characterize the efficient influence function for $\theta_{1,1}$ in Theorem 3.1. The proof of Theorem 3.1 follows the approach in Section 3.4 of Bickel et al. (1993). In particular, we characterize the tangent space of the set of distributions consistent with Assumptions 2.1, 2.2, and 2.4 at $P$, which are needed for the identification of $\theta_{1,1}$, and verify that a conjectured efficient influence function is a pathwise derivative of $\theta_{1,1}$ at $P$ and is an element of the tangent space.

**Theorem 3.1.** Under Assumptions 2.1, 2.2, and 2.4 and if treatment is observed in the observational sample, the efficient influence function for the parameter $\theta_{1,1}$ is given by

$$
\psi_{1,1}(b) = \frac{g}{\pi} \left( \frac{w(y - \mu_1(s,x))}{q_1(s,x)} + \mu_1(x) - \theta_{1,1} \right) + \frac{1 - g}{\pi} \left( \frac{\gamma(x) \cdot w(\mu_1(s,x) - \mu_1(x))}{e(x)} \right). \tag{3.2}
$$

**Corollary 3.1.** An analogous argument shows that, under Assumptions 2.1, 2.2, and 2.4 and if treatment is observed in the observational sample, the efficient influence function for $\theta_{0,1}$ is given by

$$
\psi_{0,1}(b) = \frac{g}{\pi} \left( \frac{(1 - w)(y - \mu_0(s,x))}{q_0(s,x)} + \mu_0(x) - \theta_{0,1} \right) + \frac{1 - g}{\pi} \left( \frac{\gamma(x) \cdot (1 - w)(\mu_0(s,x) - \mu_0(x))}{1 - e(x)} \right). \tag{3.3}
$$

Thus, the efficient influence function for $\tau_1$ is given by

$$
\psi_1(b) = \frac{g}{\pi} \left( \frac{w(y - \mu_1(s,x))}{q_1(s,x)} - \frac{(1 - w)(y - \mu_0(s,x))}{q_0(s,x)} + (\mu_1(x) - \mu_0(x) - \tau_1) \right) + \frac{1 - g}{\pi} \left( \frac{\gamma(x) \cdot (w(\mu_1(s,x) - \mu_1(x))}{e(x)} - \frac{(1 - w)(\mu_0(s,x) - \mu_0(x))}{1 - e(x)} \right). \tag{3.4}
$$

Knowledge of the efficient influence function allows us to compute the semiparametric efficiency bound for $\tau_1$. That is, we can obtain an expression giving a lower bound for the asymptotic variance of any sequence of regular estimators for $\tau_1$.  

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**Table 1**

*List of Nuisance Parameters for Section 3.1*

Notes: Table 1 presents the notation, definition, and interpretation of the nuisance parameters that appear in influence functions stated in Theorem 3.1 and Corollary 3.1. Some parameters are defined using the complete data $P_s, E_{P_s}$; however, under Assumptions 2.1, 2.2, and 2.4, all of the following parameters can be written in terms of functionals of the observed distribution, using $P$ and $E_P$.  

\[\text{List of Nuisance Parameters for Section 3.1}\]

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Corollary 3.2. Under Assumptions 2.1, 2.2, and 2.4 and if treatment is observed in the observational sample, the semiparametric efficiency bound for $\tau_1$ is given by

$$V^* = \Var(\psi_1(B)) = E_P \left[ \frac{\gamma(X)}{\pi^2} \left( \frac{\sigma_1^2(S,X)}{q_1(S,X)} + \frac{\sigma_0^2(S,X)}{q_0(S,X)} \right) + (\mu_1(X) - \mu_0(X) - \tau)^2 + C_0(S,X) + C_1(S,X) \right], \quad (3.5)$$

where

$$\sigma_w^2(s,x) = E_P [(Y(w) - \mu_w(s,x))^2 \mid S = s, X = x]$$

and

$$C_w(s,x) = \frac{\gamma(x) (\mu_w(s,x) - \mu_w(x))^2}{1 - \gamma(x) e(x)^w(1 - e(x))^{1-w}}.$$  

Remark 3.1. Equation (A.11) in the proof of Theorem 3.1 expresses the efficient influence function $\psi_{1,1}(b_1)$ for $\theta_{1,1}$ as a score function for an arbitrary parametric submodel of the general semiparametric model of interest. As this expression is not a function of $e'(X)$, the pathwise derivative of $e(X)$ along a one-parameter submodel evaluated at zero, knowledge or restriction of the propensity score $e(X)$ does not change the efficient influence function for this parameter. That is, $\psi_{1,1}(b)$ continues to be the efficient influence function for $\theta_{1,1}$ in a model where $e(X)$ known. As a direct consequence, if the propensity score is known, then $V^*$ remains the minimum asymptotic variance for any sequence of regular estimators for $\tau_1$. This result echos an analogous result for the average treatment effect under unconfounded treatment given in Hahn (1998).

Remark 3.2. The efficient influence function $\psi_{1,1}(b)$ for $\theta_{1,1}$ possesses a "double-robust" structure that is prevalent in causal inference and missing data problems (inter alia, Kang and Schafer, 2007; Bang and Robins, 2005). Observe that $\psi_{1,1}(b)$ can be expressed a function $f(\cdot)$ of five nuisance parameters $\mu_1(s,x), \mu_1(x), q_1(s,x), e(x),$ and $\gamma(x),$ defined in Table 1, by

$$\psi_{1,1}(b) = f(b, \mu_1(\cdot,\cdot), \mu_1(\cdot), q_1, e, \gamma).$$

The mean-zero property of $\psi_{1,1}(b)$ is maintained even if some of these nuisance functions are misspecified. In particular, if the arbitrary measurable functions $\tilde{\mu}_1(\cdot, \cdot)$ and $\tilde{\mu}_1(\cdot)$ replace the conditional means $\mu_1(\cdot,\cdot), \mu_1(\cdot),$ then

$$E_P [f(B, \tilde{\mu}_1(\cdot,\cdot), \tilde{\mu}_1(\cdot), q_1, e, \gamma)] = 0.$$  

Similarly, if the arbitrary measurable functions $\tilde{q}_1, \tilde{e},$ and $\tilde{\gamma}$ replace the propensities $q_1, e,$ and $\gamma,$ then

$$E_P [f(B, \mu_1(\cdot,\cdot), \mu_1(\cdot), \tilde{q}_1, \tilde{e}, \tilde{\gamma})] = 0.$$  

This result echos an analogous double-robust property of the efficient influence function of the average treatment effect under unconfounded treatment (Scharfstein, Rotnitzky and Robins, 1999), in which the mean-zero property of the efficient influence function is robust to misspecification of the the conditional means of the outcome variables and the propensity score separately.
Next, we demonstrate that efficient influence function $\psi_1(b)$ expressed in Corollary 3.1 is, in fact, the unique influence function for $\tau_1$ in the model under consideration. This property—uniqueness of an influence function for a statistical functional in a semiparametric model—has been termed local just-identification by Chen and Santos (2018).

**Theorem 3.2.** Under Assumptions 2.1, 2.2, and 2.4 and if treatment is observed in the observational sample, then $\psi_1(b)$ is the unique influence function for $\tau_1$.

**Remark 3.3.** Let $P \subset M_\nu$ be the set of probability distributions that satisfy Assumptions 2.1, 2.2, and 2.4. In the terminology of Chen and Santos (2018), Theorem 3.2 demonstrates that $P$ is locally just-identified by $P$. As a result, by Theorem 3.1 of Chen and Santos (2018), all regular and asymptotically linear estimators of $\tau_1$ are first-order equivalent. In particular, there are no regular and asymptotically linear estimators of $\tau_1$ that have smaller asymptotic variances than others. Moreover, again by Theorem 3.1 of Chen and Santos (2018), the model $P$ does not have any locally testable restrictions in the sense that there are no specification tests of the assumptions with nontrivial local asymptotic power.

**Remark 3.4.** An immediate consequence of Theorem 3.2 is that the semiparametric estimators for the long-term treatment effect $\tau_1$ proposed in Athey et al. (2020a) are first-order equivalent, and hence semiparametrically efficient, so long as they are regular and asymptotically linear. Nevertheless, it is instructive to verify the semiparametric efficiency of a particular estimator proposed therein whose form is amenable to an alternative approach of analysis. In particular, Athey et al. (2020a) propose a weighting estimator that is similar to

$$\hat{\tau}_{weight} = \frac{1}{n} \sum_{i=1}^{n} \frac{GWY}{\pi q_1(S,X)} - \frac{G(1 - W)Y}{\pi \hat{q}_0(S,X)}.$$

We focus our analysis on

$$\hat{\theta}_{1,weight} = \frac{1}{n} \sum_{i=1}^{n} \frac{GWY}{\pi q_1(S,X)},$$

which can be viewed as a semiparametric generalized method of moments (GMM) estimator with the moment function

$$m(b, \theta, \tilde{q}_1) = \frac{gwy}{\tilde{q}_1(s,x)} - \theta_{1,1},$$

for $\tilde{q}_1(s,x) = \pi q_1(s,x)$.

Applied to our setting, Theorem 2.1 of Newey (1994) demonstrates that the choice of estimator of the nuisance parameter $\tilde{q}(s,x)$ has no impact on the asymptotic variance of the estimator, so long as the resulting two-step GMM estimator is regular and asymptotically linear (and hence $\sqrt{n}$-consistent). Moreover, expressions (3.8) and (3.10) of Newey (1994) demonstrate that the influence function for $\hat{\theta}_{1,weight}$ takes the form

$$\psi_{1,weight}(b) = m(b, \theta_{1,1}, \tilde{q}_1) + \alpha(b),$$

where the Riesz representer $\alpha(b)$ is such that

$$\mathbb{E}_P[\alpha(B)\ell'(B)] = M'(P, \theta_{1,1}, q_1), \quad (3.6)$$

6The estimator proposed in the Athey et al. (2020a) has self-normalizing weights and targets $\mathbb{E}[Y(1) - Y(0)]$ rather than $\mathbb{E}[Y(1) - Y(0) | G = 1]$. 
where we recall that the right-hand side of this expression is the pathwise derivative of the statistical functional

$$M(P, \theta_1, \tilde{q}_1) = \mathbb{E}_P[m(B, \theta_1, \tilde{q}_1)]$$

along an arbitrary parametric submodel of the maintained semiparametric model evaluated at zero. In Appendix A.4, we verify that

$$\alpha(b) = \theta_{1,1} - \frac{g}{\pi} \left( \frac{w\mu_1(s,x)}{q_1(s,x)} - (\mu_1(x) - \theta_{1,1}) \right) + \frac{1 - g}{\pi} \left( \frac{\gamma(x) w(\mu_1(s,x) - \mu_1(x))}{1 - \gamma(x) e(x)} \right)$$  \hspace{1cm} (3.7)

satisfies the Riesz representation condition (3.6). Thus, we find that the influence function for this estimator

$$\psi_{1,\text{weight}}(b) = m(b, \theta_1, \tilde{q}_1) + \alpha(b) = \psi_{1,1}(b)$$

agrees with the efficient influence function computed in Theorem 3.1, and hence that $\hat{\tau}_{\text{weight}}$ is semiparametrically efficient if it is regular and asymptotically linear.

**Remark 3.5.** Another approach to estimation of $\tau_1$, studied in e.g., Chernozhukov et al. (2016) and Chernozhukov et al. (2018), proceeds by using the mean-zero property of the efficient influence function as a semiparametric moment condition by directly estimating the five nuisance parameters $\mu_w(S,X)$, $\mu_w(X)$, $e(X)$, $\gamma(X)$, and $q_w(S,X)$, summarized in Table 1, and applying an appropriate form of sample-splitting. The double-robustness property of the efficient influence function outlined in Remark 3.2 suggests that the consistency of such an estimator will be robust to misspecification of the condition means $\mu_w(S,X)$ and $\mu_w(X)$ or the propensity scores $e(X)$, $\gamma(X)$, $q_w(S,X)$, respectively. Nevertheless, the nuisance parameters in the efficient influence function may be cumbersome to estimate in practice. In particular, by expression (3.1), estimation of $q_w(S,X)$ involves either the estimation of density ratios or odds ratios.

As noted in Remark 3.3, all regular and asymptotically linear estimators of $\tau_1$ have the same asymptotic variance. Thus, a nonparametric estimator constructed by directly estimating the nuisance parameters that appear in the efficient influence function would have first-order asymptotics that are matched by computationally simpler estimators like $\hat{\tau}_{\text{weight}}$. Nevertheless, the more complicated estimator may accommodate relaxed rate conditions for the estimation of nonparametric components while maintaining $\sqrt{n}$-normality.

3.2. Treatment is Not Observed in the Observational Sample. Next, we consider the case in which treatment is not observed in the observational sample, where we now maintain Assumptions 2.1 to 2.3 and 2.5. Again, we present the efficient influence function and semiparametric efficiency bound for $\tau_1$, defined in Equation (2.1), and give analogous results for $\tau_0$ in Appendix B.

The change in maintained identifying assumptions necessitates the definition of several additional nuisance functions, which are collected in Table 2. Let

$$\mu(s, x) = \mathbb{E}_P[Y \mid S, X, G = 1]$$

denote the conditional expectation of the long-term outcome in the observational sample conditional on the short-term outcomes and covariates. Observe that $\mu(s, x)$ does not have
a \( w \)-subscript and does not condition on \( W \)—which is natural as \( W \) is not observed in the observational sample. Athey et al. (2020b) term \( \mu(s, x) \) the surrogate index for \( Y \). Additionally, let

\[
\begin{align*}
    r(s, x) &= P(W = 1 \mid S = s, G = 0, X = x) \quad \text{and} \quad t(s, x) = P(G = 0 \mid S = s, X = x)
\end{align*}
\]

denote the probability of treatment in the experimental sample conditional on the short-term outcome and covariates and the probability of being in the experimental sample conditional on the short-term outcomes and the covariates, respectively. Athey et al. (2020b) terms \( r(s, x) \) and \( t(s, x) \) as the surrogate score and sampling score, respectively.

We begin again by characterizing the efficient influence function for \( \theta_{1,1} \), following the approach considered in Section 3.4 of Bickel et al. (1993), with an immediate corollary giving the efficient influence functions for \( \theta_{0,1} \) and \( \tau_1 \).

**Theorem 3.3.** Under Assumptions 2.1 to 2.3 and 2.5 and if treatment is not observed in the observational sample, the efficient influence function for the parameter \( \theta_{1,1} \) is given by

\[
\begin{align*}
    \xi_{1,1}(b) &= \frac{g}{\pi} \left( \frac{\gamma(x)}{1 - \gamma(x)} \frac{r(s, x)t(s, x)(y - \mu(s, x))}{1 - t(s, x)} + (\mu_1(x) - \theta_{1,1}) \right) \\
    &\quad + \frac{1 - g}{\pi} \left( \frac{w(\mu(s, x) - \mu_1(x))}{e(x)} \right).
\end{align*}
\]

**Corollary 3.3.** An analogous argument shows that, under Assumptions 2.1 to 2.3 and 2.5 and if treatment is not observed in the observational sample, the efficient influence function for \( \theta_{0,1} \) is given by

\[
\begin{align*}
    \xi_{0,1}(b) &= \frac{g}{\pi} \left( \frac{\gamma(x)}{1 - \gamma(x)} \frac{(1 - r(s, x))t(s, x)(y - \mu(s, x))}{1 - t(s, x)} + (\mu_0(x) - \theta_{0,1}) \right) \\
    &\quad + \frac{1 - g}{\pi} \left( \frac{(1 - w)(\mu(s, x) - \mu_0(x))}{1 - e(x)} \right).
\end{align*}
\]

Thus, the efficient influence function for \( \tau_1 \) is given by

\[
\begin{align*}
    \xi_1(b) &= \frac{g}{\pi} \left( \frac{\gamma(x)}{1 - \gamma(x)} \frac{t(s, x) - e(x)}{1 - t(s, x)} \frac{r(s, x) - e(x)}{e(x)(1 - e(x)) (y - \mu(s, x)) + (\mu_1(x) - \mu_0(x) - \tau_1)} \right) \\
    &\quad + \frac{1 - g}{\pi} \left( \frac{w(\mu(s, x) - \mu_1(x))}{e(x)} - \frac{(1 - w)(\mu(s, x) - \mu_0(x))}{1 - e(x)} \right).
\end{align*}
\]
Derivation of the efficient influence function again allows us to compute the semiparametric efficiency bound for $\tau_1$.

**Corollary 3.4.** Under Assumptions 2.1 to 2.3 and 2.5 and if treatment is not observed in the observational sample, the semiparametric efficiency bound for $\tau_1$ is given by

$$V^{**} = \text{Var}(\xi_1(B)) = \mathbb{E}_P\left[\frac{1}{\pi^2} \left( \frac{\gamma^3(X)}{(1-\gamma(X))^2} \frac{t^2(s,x)}{(1-t(s,x))^2} \frac{(r(s,x) - e(x))^2}{e^2(x)(1-e(x))^2} \sigma^2(S,X) \right. \right.$$  
$$\left. + \gamma(X)(\mu_1(X) - \mu_0(X) - \tau)^2 + D_0(S,X) + D_1(S,X) \right]\right),$$  \hspace{1cm} (3.11)

where

$$\sigma^2(s,x) = \mathbb{E}_P[(Y - \mu(s,x))^2 | S = s, X = x]$$

and

$$D_w(s,x) = (1 - \gamma(x)) \frac{(\mu(s,x) - \mu_w(x))^2}{e(x)w(1-e(x))^{1-w}}.$$

**Remark 3.6.** Equation (A.32) in the proof of Theorem 3.3 expresses the efficient influence function $\xi_{1,1}(\hat{b}_1)$ for $\theta_{1,1}$ as a score function for an arbitrary parametric submodel of the general semiparametric model of interest. In this case, the expression is not a function of both $e'(X)$ and $p'(S | X, G = 1)$, the pathwise derivatives of the propensity score and the conditional distribution of the short-term outcome given covariates on the observational sample evaluated at zero. Thus, the efficient influence function for $\theta_{1,1}$ does not change if either of these nuisance parameters are known. Consequently, if either of these nuisance parameters admits known restrictions, then $V^{**}$ remains the minimum asymptotic variance of any sequence of regular estimators for $\tau_1$.

**Remark 3.7.** The efficient influence function $\xi_{1,1}(b)$ for $\theta_{1,1}$ again possesses a double-robustness property. Observe that $\xi_{1,1}(b)$ can be written as a function $g(\cdot)$ of six nuisance functions $\mu(s,x), \mu_1(x), r(s,x), e(x), t(s,x)$, and $\gamma(x)$, defined in Table 2, as

$$\xi_{1,1}(b) = g(b, \mu(\cdot), \mu_1(\cdot), r, e, t, \gamma).$$

The mean-zero property of $\xi_{1,1}(b)$ is maintained if the conditional means $\mu(s,x)$ and $\mu_1(x)$ or the propensity type objects $r(s,x), e(x), t(s,x)$, and $\gamma(x)$ are misspecified, respectively, in the same manner outlined in Remark 3.2.

Lastly, again, we demonstrate that the efficient influence function $\xi_1(b)$ presented in Corollary 3.3 is the unique influence function for $\tau_1$ in the model under consideration. That is, again, $\tau_1$ is locally just-identified in the sense of Chen and Santos (2018) and so all regular and asymptotically linear estimators for $\tau_1$ are first-order equivalent.

**Theorem 3.4.** Under Assumptions 2.1 to 2.3 and 2.5 and if treatment is not observed in the observational sample, then $\xi_1(b)$ is the unique influence function for $\tau_1$. 
4. Conclusion. We consider the statistical structure of semiparametric estimation of long-term treatment effects through the combination of short-term experimental and long-term observational datasets. We derive efficient influence functions and calculate the corresponding semiparametric efficiency bounds for several related models of this problem with different data structures. We find in each case that the efficient influence function is the unique influence function for the long-term treatment effect, and that therefore all regular and asymptotically linear estimators of this parameter are first-order equivalent.

There are many important open questions remaining in the estimation of long-term treatment effects. In practice, particularly important unresolved issues concern the choice of valid and informative short-term outcomes and the assessment of the validity of identifying assumptions concerning the relationship between short-term and long-term outcomes. Useful extensions of the models that we consider include the incorporation of instruments and continuous treatments, consideration of some degree of unconfounded treatment in the observational sample, and the consideration of an alternative asymptotic frameworks in which the number of short-term outcomes grows with the sample size—and perhaps are selected with some data-driven procedure.

REFERENCES


APPENDIX A: PROOFS FOR THEOREMS PRESENTED IN THE MAIN TEXT

A.1. Proof of Theorem 2.1. Unless otherwise noted, we drop the dependence on the individual \( i \). It will suffice to consider the parameter

\[ \theta_{1,1} = \mathbb{E}_{P_s}[Y(1) \mid G = 1] , \]

as the point-identification of \( \theta_{1,0} = \mathbb{E}[Y(0) \mid G = 1] \) will follow by an analogous argument. First, we consider the case that treatment is observed in the observational sample, where we operate under Assumptions 2.1 to 2.3 and 2.5. Define the functions

\[ \mu_1(s, x) = \mathbb{E}_P[Y \mid S = s, X = x, W = 1, G = 1] \]

\[ = \mathbb{E}_{P_s}[Y(1) \mid S(1) = s, X = x, W = 1, G = 1] \]

and observe that \( \mu_1(s, x) \) is identified from the observational sample and \( \mu_1(x) \) is identified from the experimental sample given the identification of \( \mu_1(s, x) \). Observe that

\[ \mathbb{E}_{P_s}[Y(1) \mid G = 1] \]

\[ = \mathbb{E}_{P_s}[\mathbb{E}_{P_s}[Y(1) \mid X, G = 1] \mid G = 1] \]

\[ = \mathbb{E}_{P_s}[\mathbb{E}_{P_s}[Y(1) \mid S(1), X, G = 1] \mid X, G = 0] \mid G = 1 \]  

(Assumption 2.2)

\[ = \mathbb{E}_{P_s}[\mathbb{E}_{P_s}[Y(1) \mid S(1), X, G = 1] \mid X, G = 0] \mid G = 1 \]

\[ = \mathbb{E}_{P_s}[\mathbb{E}_{P_s}[Y(1) \mid S(1), X, G = 1] \mid X, G = 0] \mid G = 1 \]  

(Assumption 2.4)

\[ = \mathbb{E}_{P_s}[\mathbb{E}_{P_s}[\mu_1(S(1), X)] \mid X, G = 0] \mid G = 1 \]

\[ = \mathbb{E}_{P_s}[\mathbb{E}_{P_s}[\mu_1(S(1), X)] \mid X, G = 0] \mid G = 1 \]  

(Assumption 2.1)

\[ = \mathbb{E}_{P_s}[\mu_1(X) \mid G = 1] . \]

Hence, \( \tau_1 \) is identified as \( \mu_1(x) \) is identified.

Next, we consider the case that treatment is not observed in the observational sample, where we operate under Assumptions 2.1 to 2.3 and 2.5. Define the function

\[ \mu(s, x) = \mathbb{E}_P[Y \mid S = s, X = x, G = 1] \]

and observe that \( \mu(s, x) \) is identified from the observational sample. Observe that

\[ \mathbb{E}_{P_s}[Y(1) \mid G = 1] = \mathbb{E}_{P_s}[\mathbb{E}_{P_s}[Y(1) \mid X, G = 1] \mid G = 1] \]

\[ = \mathbb{E}_{P_s}[\mathbb{E}_{P_s}[Y(1) \mid X, G = 0] \mid G = 1] \]  

(Assumption 2.2)

\[ = \mathbb{E}_{P_s}[\mathbb{E}_{P_s}[Y(1) \mid X, G = 0] \mid G = 1] \]

\[ = \mathbb{E}_{P_s}[\mu_1(X) \mid G = 1] , \]  

(Assumption 2.1)

and that

\[ \mathbb{E}_{P_s}[Y(1) \mid X, W = 1, G = 0] \]

\[ = \mathbb{E}_P[Y \mid X, W = 1, G = 0] \]

\[ = \mathbb{E}_P[\mathbb{E}_P[Y \mid S, X, W = 1, G = 0] \mid X, W = 1, G = 0] \]

\[ = \mathbb{E}_P[\mathbb{E}_P[Y \mid S, X, W = 1, G = 1] \mid X, W = 1, G = 0] \]  

(Assumption 2.3)

\[ = \mathbb{E}_P[\mathbb{E}_P[Y \mid S, X, G = 1] \mid X, W = 1, G = 0] \]  

(Assumption 2.5)

\[ = \mathbb{E}_P[\mu(S, X) \mid X, W = 1, G = 0] . \]  

(Assumption 2.5)

It is clear that \( \mathbb{E}_P[\mu(S, X) \mid X, W = 1, G = 0] \) is identified from the experimental sample, and that therefore \( \tau_1 \) is identified.
A.2. Proof of Theorem 3.1. Unless otherwise noted, we drop the dependence on the individual $i$. We omit the subscript $w$ for each of the parameters displayed in Table 1, as only the case $W = 1$ is relevant for our estimand $\theta_{1,1}$. The presentation of the proof will be somewhat constructive, as we hope to convey the rationale behind the form of the efficient influence function, for which we have had to take a few educated guesses. The reader can of course verify Theorem 3.1 directly by using the stated form of the influence function and checking a series of conditions that we lay out.

The argument proceeds as follows. First, we factorize the density of the observed data and relate each factor to conditional densities of the complete data distribution. Second, for an arbitrary smooth parametric submodel, we characterize the tangent space of the distribution of the observed data as a linear space of mean-zero square-integrable functions. Third, we conjecture a functional form that explicitly resides in the tangent space, where we note that the efficient influence function is the unique Riesz representer of the target statistical functional $\theta_{1,1}$ in the tangent space for the pathwise derivatives (Chapter 25 of Van der Vaart, 2000). Fourth, we state the conditions a Riesz representer of the pathwise derivative of $\theta_{1,1}$ necessarily satisfies, and we verify that the conjectured efficient influence function satisfies these conditions.

Density Factorization. This subsection establishes the factorization of the density of the complete data distribution, displayed below in (A.1). Consider the random variable

$$A_1 = (Y(1), S(1), W, G, X) \equiv (Y^*, S^*, W, G, X)$$

where we observe $B_1 = (WGY^*, W^*S, W, G, X)$. Let $Y = WGY^*$ and $S = WS^*$. Recall that the distribution of the complete data is denoted by $P_\ast$, with density $p_\ast$ with respect to $\nu$, and that the distribution of the observed data is denoted by $P$, with density $p$ with respect to $\nu$. Observe that the densities under the complete data distribution and the observed data distribution agree when data isn’t missing. That is, for all $(y, s) \in \mathcal{Y} \times \mathcal{S}$ and events $E \in \mathcal{A}$, we have that

$$p_\ast(y \mid W = 1, G = 1, E) = p(y \mid W = 1, G = 1, E)$$

and

$$p_\ast(s \mid W = 1, E) = p(s \mid W = 1, E)$$

and that for all events $E$ measurable with respect to the $\sigma$-algebra generated by $(W, G, X)$, we have that $P(E) = P_\ast(E)$.

By Assumptions 2.1, 2.2, and 2.4, the density of the observed data $p$ admits the factorization

$$p(b_1) = p(y \mid G = 1, X = x, W = 1, S = s)^{w g} p(s, W = 1 \mid G = 1, X = x)^{w g}$$

$$p(s \mid G = 0, X = x, W = 1)^{w (1 - g)}$$

$$P(W = w \mid G = 0, X = x)^{1 - g} P(W = 0 \mid G = 1, X = x)^{(1 - w) g} P(G = g \mid X = x) p(x).$$

Each of the terms of this expression can be rewritten in terms of the density of the complete data $p_\ast$. In particular, we have that the first term can be written

$$p(y \mid G = 1, X = x, W = 1, S = s) = p_\ast(y \mid G = 1, X = x, W = 1, S = s)$$

$$= p_\ast(y \mid x, s), \quad (\text{Assumption 2.4 and Assumption 2.2})$$

the second and third terms can be written

$$p(s, W = 1 \mid G = 1, X = x) = p_\ast(S^* = s, W = 1 \mid G = 1, X = x)$$

$$= p_\ast(W = 1 \mid S^* = s, G = 1, X = x) p_\ast(S^* = s \mid G = 1, X = x)$$

$$= q(s, x) p_\ast(s \mid x) \quad (\text{Assumption 2.2})$$

and

$$p(s \mid G = 0, X = x, W = 1) = p_\ast(S^* = s \mid G = 0, X = x)$$

$$= p_\ast(s \mid x), \quad (\text{Assumption 2.1})$$

(\text{Assumption 2.2})
the fourth term can be written
\[ P(W = w \mid G = 0, X = x) = e(x)^w (1 - e(x))^{1-w}, \] (Assumption 2.1)
and finally the fifth term can be written
\[ P(W = 0 \mid G = 1, X = x) = P_*(W = 0 \mid G = 1, X = x) \]
\[ = \int P_*(W = 0 \mid G = 1, X = x, S = s) \, dP_*(s \mid x) \]
\[ = \int (1 - q(s, x)) p_*(s \mid x) \, ds. \]

As a result, we obtain a factorization of the density of the observed data in terms of conditional distributions of the complete data
\[
p(b_1) = p_*(y \mid x, s)^w p_*(s \mid x)^w q(s, x) w g \left( \int (1 - q(s, x)) p_*(s \mid x) \, ds \right)^{(1-w)g} \]
\[
\gamma(x)^g (1 - \gamma(x))^{1-g} (e(x)^w (1 - e(x))^{1-w})^{1-g} p(x). \] (A.1)

Characterization of the Tangent Space. Let \( \mathcal{P} \) be any regular parametric submodel of \( \mathcal{M}_\nu \) indexed by \( \eta \in \mathbb{R} \), with densities \( p_0 := dP_0/d\nu \) with \( p_0 = dP/d\nu \) and such that Assumptions 2.1, 2.2, and 2.4 hold for each \( P_\eta \in \mathcal{P} \). Let the subscript \( * \) distinguish log densities of the complete data from log densities of the observed data. By the factorization (A.1), we obtain the score
\[
\ell'(b_1) = wg \cdot \ell'_*(y \mid x, s) + w \cdot \ell'_*(s \mid x) + \ell'_*(g, x) + wg \cdot q'(s, x)
- g(1 - w) \left( \frac{E_{P_*}[q(S^*, x)\ell'_*(S^* \mid x) \mid G = 0, W = 1, X = x]}{1 - E_{P_*}[q(S^*, x) \mid G = 0, W = 1, X = x]} \right)
+ g \left( \frac{w \cdot q'(s, x)}{q(s, x)} - (1 - w) \frac{E_{P_*}[q'(S^*, x) \mid G = 0, W = 1, X = x]}{1 - E_{P_*}[q(S^*, x) \mid G = 0, W = 1, X = x]} \right)
+ (1 - g) e'(x) \left( \frac{w - e(x)}{e(x)(1 - e(x))} \right). \] (A.2)

Thus, the tangent space \( \mathcal{T} \) is given by mean-square closure of the linear span of the functions
\[
s(b_1) = wg \cdot s_1(y \mid x, s) + w \cdot s_2(s \mid x) + s_3(g, x) + wg \cdot s_4(s, x)
- g(1 - w) \left( \frac{E_{P_*}[q(S^*, x)s_2(S^* \mid x) \mid G = 0, W = 1, X = x]}{1 - E_{P_*}[q(S^*, x) \mid G = 0, W = 1, X = x]} \right)
+ g \left( \frac{w \cdot s_4(s, x)}{q(s, x)} - (1 - w) \frac{E_{P_*}[s_4(S^*, x) \mid G = 0, W = 1, X = x]}{1 - E_{P_*}[q(S^*, x) \mid G = 0, W = 1, X = x]} \right)
+ (1 - g) \left( s_5(x) \frac{w - e(x)}{e(x)(1 - e(x))} \right),
\]
where the functions \( s_1 \) through \( s_5 \) range over the space of mean-zero and square integrable functions that satisfy the restrictions
\[
E_P[s_1(Y \mid S, X) \mid W = 1, G = 1, S, X] = E_{P_*}[s_1(Y^* \mid s, x) \mid S^* = s, X = x] = 0,
E_P[s_2(S \mid X) \mid W = 1, G = 0, X] = E_{P_*}[s_2(S^* \mid x) \mid X = x] = 0,
E_P[s_3(G, X)] = 0.
\]

Pathwise Differentiability Conditions. In the following calculations, it is often easier to work with the complete data distribution. To that end, we note that for an arbitrary measurable function
h, we have that certain conditional means of the observed distribution equal certain conditional means of the complete data distribution:

\[ E_P[h(Y, s, x) \mid W = 1, G = 1, X = x, S = s] = E_{P_1}[h(Y^*, s, x) \mid G = 1, X = x, S^* = s], \]

\[ E_P[h(S, x) \mid G = 0, W = 1, X = x] = E_{P_1}[h(S^*, x) \mid G = 0, X = x] \]

\[ = E_{P_1}[h(S^*, x) \mid G = 1, X = x], \]

\[ E_P[h(X) \mid G = 1] = E_{P_1}[h(X) \mid G = 1], \]

and, as a result, for h,

\[ E_P[E_P[h(Y, S, X) \mid W = 1, G = 1, X, S \mid W = 1, G = 0, X] \mid G = 1] \]

\[ = E_{P_1}[h(Y^*, S^*, X) \mid G = 1] = E_{P_1} \left[ \frac{G}{\pi} h(Y^*, S^*, X) \right]. \]

Note that \( \theta_{1,1} \) is such a functional with \( h(y, s, x) = y \).

Observe that the parameter of interest can be written

\[ \theta_{1,1} = E_{P_1}[Y^* \mid G = 1] \]

\[ = E_P[E_P[Y \mid W = 1, G = 1, X, S \mid W = 1, G = 0, X] \mid G = 1] \]

\[ = \int_X \int_S \int_Y \frac{y}{\pi} p_s(s \mid y)p(y \mid s,x)p(x \mid G = 1) d\nu_Y(y) d\nu_s(s) d\nu_X(x), \]

where \( \mu_D(d) \) denotes the marginal distribution of \( D \) under \( \nu \). The pathwise derivative of this parameter at 0 on \( \mathcal{P} \) is given by

\[ \theta'_{1,1} = E_{P_1}[Y^* \ell'(Y^* \mid S^*, X) \mid G = 1] \]

\[ + E_{P_1}[Y^* \ell'(S^* \mid X) \mid G = 1] + E_{P_1}[Y^* \ell'(X, G = 1) \mid G = 1]. \] (A.3)

Now, observe that

\[ G \cdot \ell'_s(X, G = 1) = G \cdot \ell'_s(X, G) - G \cdot \ell'_s(G = 1), \]

and that

\[ \ell'_s(G = 1) = E_{P_1}[\ell'_s(X, G) \mid G = 1]. \]

Thus, each of the terms in (A.3) can be written in terms of conditional scores by

\[ E_{P_1}[Y^* \ell'(Y^* \mid S^*, X) \mid G = 1] = E_{P_1}[\pi^{-1}GY^* \ell'_s(Y^* \mid S^*, X)], \]

\[ E_{P_1}[Y^* \ell'(S^* \mid X) \mid G = 1] = E_{P_1}[\pi^{-1}GY^* \ell'_s(S^* \mid X)], \] and

\[ E_{P_1}[Y^* \ell'(X, G = 1) \mid G = 1] = E_{P_1}[\pi^{-1}GY^* \ell'_s(X, G = 1)] \]

\[ = E_{P_1}[\pi^{-1}GY^* \ell'_s(X, G)] - E_{P_1}[\pi^{-1}GY^*] \ell'_s(G = 1) \]

\[ = E_{P_1}[\pi^{-1}GY^* \ell'_s(X, G)] - \theta_{1,1} \cdot E_{P_1}[\ell'_s(X, G) \mid G = 1], \]

respectively.

An influence function for \( \theta_{1,1} \) is a mean-zero and square-integrable function \( \tilde{\psi}_{1,1}(B_1) \) that satisfies the condition

\[ \theta'_{1,1} = E_P[\tilde{\psi}_{1,1}(B_1) \ell'(B_1)]. \] (A.4)

Hence, by (A.2) and (A.3), in order to establish that a mean-zero and square-integrable function \( \tilde{\psi}(B_1) \) is an influence function for \( \theta_1 \), it suffices to show that the terms in the right-hand side of (A.4) match the terms in the right-hand side of (A.3):

\[ \frac{1}{\pi} E_{P_1}[GY^* \ell'_s(Y^* \mid S^*, X)] = E_P \left[ \tilde{\psi}_{1,1}(B_1)GW \ell'_s(Y \mid S, X) \right], \] (A.5)
\[
\frac{1}{\pi} E_P [GY^* \ell_y^*(S^* | X)] = E_P \left[ \psi_{1,1}(B_1) \left( W \ell_y^*(S | X) - G(1 - W) \int q(s, X)p_y'(s | X) d\nu_S(s) \right) \right], \quad (A.6)
\]
\[
E_P [\tilde{\psi}_{1,1}(B_1) \ell_y^*(G, X)] = \frac{1}{\pi} E_P [GY^* \ell_y^*(G, X)] - \theta_{1,1} \cdot E_P [\ell_y^*(G, X) | G = 1], \quad (A.7)
\]
\[
0 = E_P \left[ \tilde{\psi}_{1,1}(B_1)G \left( W \frac{q'(S, X)}{q(S, X)} - (1 - W) \int q'(S, X)p_y(s | X) d\nu_S(s) \right) \right], \quad (A.8)
\]
\[
0 = E_P \left[ \tilde{\psi}_{1,1}(B_1)(1 - G) \frac{W - e(X)}{e(X)(1 - e(X))} e(X) \right]. \quad (A.9)
\]

**Conjectured Efficient Influence Function.** We begin by considering an element of the tangent space. For the unspecified functions \( \phi(s, x) \) and \( \varphi(x) \), consider the choices
\[
s_1(y, s, x) = \frac{y - \mu(s, x)}{\pi q(s, x)}, \quad s_2(s, x) = \frac{1}{e(x)(1 - \gamma(x))},
\]
\[
s_3(g, x) = g\varphi(x), \quad s_4(s, x) = -q(s, x) s_2(s, x), \quad \text{and}
\]
\[
s_5(x) = 0. \quad (A.10)
\]
These choices correspond to the following conjectured efficient influence function
\[
\psi_{1,1}(b_1) = gw \cdot s_1(y, s, x) + w \cdot s_2(s, x) - gw \cdot s_2(s, x) + g\varphi(x)
\]
\[
= gw \cdot s_1(y, s, x) + (1 - g)w \cdot s_2(s, x) + g \cdot \varphi(x)
\]
\[
= gw(y - \mu(s, x)) + \frac{(1 - g)w}{1 - \gamma(x)e(x)} \phi(s, x) + g\varphi(x). \quad (A.11)
\]
In particular, \( s_4 \) is chosen so that certain terms cancel to construct the term \((1 - g)ws_2(s, x)\), which does not, prima facie, belong in the tangent space \((A.2)\).

In order to ensure that \( \psi_{1,1}(b_1) \) belongs to the tangent space, we require
\[
E_P[\phi(S, x) | W = 1, G = 0, X = x] = E_P[\phi(S^*, x) | X = x] = 0, \quad \text{and}
\]
\[
E_P[G\varphi(X)] = 0. \quad (A.12)
\]

In the sequel, we verify that the choices
\[
\phi(s, x) = \frac{\gamma(x)}{\pi} (\mu(s, x) - \mu(x)), \quad \text{and}
\]
\[
\varphi(x) = \frac{\mu(x) - \theta_{1,1}}{\pi}
\]
satisfy these conditions, and, moreover, give rise to the efficient influence function by satisfying the conditions \((A.5)-(A.9)\).

**Verifying Pathwise Differentiability Conditions.** We begin by stating and proving a useful lemma for handling terms of the form \( GW h(Y, S, X) \).

**Lemma A.1.** Under the assumptions of **Theorem 3.1**, if \( h(y, s, x) \) is any measurable function, then
\[
E_P \left[ \frac{GW}{q(S, X)} h(Y, S, X) \right] = E_P \left[ Gh(Y^*, S^*, X) \right] = E \left[ GE[h(Y^*, S^*, X) | S^*, X] \right].
\]
PROOF. Observe that
\[ \mathbb{E}_P \left[ \frac{GW}{q(S, X)} h(Y, S, X) \right] = \mathbb{E}_P \left[ \frac{W}{q(S, X)} h(Y, S, X) \mid G = 1 \right] \]
\[ = \pi \mathbb{E}_{P_s} \left[ \frac{W}{q(S^*, X)} h(Y^*, S^*, X) \mid G = 1 \right] \]
\[ = \pi \mathbb{E}_{P_s} \left[ q^{-1}(S^*, X) \mathbb{E}_{P_s} [W \mid S^*, X, G = 1] \right] \]
\[ = \mathbb{E}_{P_s} [h(Y^*, S^*, X) \mid S^*, X, G = 1] \mid G = 1 \] \hspace{1cm} (Assumption 2.4)
\[ = \mathbb{E}_{P_s} [\pi h(Y^*, S^*, X) \mid G = 1] \]
\[ = \mathbb{E}_{P_s} [G h(Y^*, S^*, X)] , \]
giving the first equality. The second equality then follows from
\[ \mathbb{E}_{P_s} [G h(Y^*, S^*, X)] = \pi \mathbb{E}_{P_s} [\mathbb{E}_{P_s} [h(Y^*, S^*, X) \mid S^*, X, G = 1] \mid G = 1] \]
\[ = \mathbb{E}_{P_s} [G \mathbb{E}_{P_s} [h(Y^*, S^*, X) \mid S^*, X, G = 1]] \]
\[ = \mathbb{E}_{P_s} [G \mathbb{E}_{P_s} [h(Y^*, S^*, X) \mid S^*, X]] , \]
completing the proof. ■

It now suffices to verify the conditions (A.5)–(A.9) for choices of \( \phi(S, X) \) and \( \varphi(X) \) that satisfy (A.12) and (A.13). The remainder of the proof verifies these conditions sequentially.

**Condition (A.5):** Observe that
\[ \mathbb{E}_P \left[ \psi(B_1) \right] \left[ G W \ell'_s(Y \mid S, X) \right] = \mathbb{E}_P \left[ \frac{GW}{\pi q(S, X)} \ell'_s(Y \mid S, X) \right] \]
\[ = \mathbb{E}_{P_s} \left[ \pi^{-1} G Y^* \ell'_s(Y^* \mid S^*, X) \right] , \hspace{1cm} \text{(Lemma A.1)} \]
as required.

**Condition (A.6):** We derive the restrictions on \( \phi(s, x) \) implied by (A.6). Observe that the inner product from (A.6) can be written
\[ \mathbb{E}_P \left[ \psi_{1,1}(B_1) \left( W \ell'_s(S \mid X) - G(1 - W) \frac{\mathbb{E}_{P_s} [q(S, X) \ell'_s(S \mid X) \mid X]}{1 - \mathbb{E}_{P_s} [q(S, X) \mid X]} \right) \right] \]
\[ = \mathbb{E}_P \left[ \frac{GW(Y - \mu(S, X)) \ell'_s(S \mid X)}{\pi q(S, X)} \right] - \mathbb{E}_P \left[ \varphi(X) G(1 - W) \frac{\mathbb{E}_{P_s} [q(S, X) \ell'_s(S \mid X) \mid X]}{1 - \mathbb{E}_{P_s} [q(S, X) \mid X]} \right] \]
\[ + \mathbb{E}_P \left[ \frac{(1 - G) W \phi(S, X)}{(1 - \gamma(X)) \psi(x)} \ell'_s(S \mid X) \right] + \mathbb{E}_P \left[ GW \varphi(X) \ell'_s(S \mid X) \right] . \]

Note that the first term can be eliminated by **Lemma A.1:**
\[ \mathbb{E}_P \left[ \frac{GW(Y - \mu(S, X)) \ell'_s(S \mid X)}{\pi q(S, X)} \right] = 0 . \]

Observe that
\[ \mathbb{E}_{P_s} [G(1 - W) \mid X] = P_s(W \mid G = 1, X) \gamma(X) = (1 - \mathbb{E}_{P_s} [q(S, X) \mid X]) \gamma(X) , \]
and thus the second term is given by
\[ \mathbb{E}_P \left[ \frac{\varphi(X) G(1 - W) \mathbb{E}_{P_s} [q(S^*, X) \ell'_s(S^* \mid X; 0) \mid X]}{1 - \mathbb{E}_{P_s} [q(S^*, X) \mid X]} \right] \]
\[ = \mathbb{E}_{P_s} \left[ \gamma(X) \varphi(X) \mathbb{E}_{P_s} [q(S^*, X) \ell'_s(S^* \mid X; 0) \mid X] \right] \]
\[ = \mathbb{E}_{P_s} \left[ \gamma(X) \varphi(X) q(S^*, X) \ell'_s(S^* \mid X) \right] . \]
Next, observe that

$$
\mathbb{E}_P[(1 - G)W \mid S^*, X] = (1 - \gamma(X))P_s(W = 1 \mid G = 0, S^*, X) = (1 - \gamma(X))\epsilon(X),
$$

and therefore the third term can be written

$$
\mathbb{E}_P \left[ \frac{(1 - G)W}{(1 - \gamma(X))\epsilon(X)} \phi(S, X)\ell'_s(S \mid X; 0) \right] = \mathbb{E}_P \left[ \phi(S^*, X)\ell'_s(S^* \mid X; 0) \right].
$$

The fourth term is given by

$$
\mathbb{E}_P[GW\varphi(X)\ell'_s(S \mid X; 0)] = \mathbb{E}_P[\gamma(X)q(S^*, X)\varphi(X)\ell'_s(S \mid X; 0)],
$$

which cancels with the second term.

Thus, the condition (A.6) is equivalent to

$$
\mathbb{E}_P[\ell'_s(S^* \mid X; 0)\phi(S^*, X)] = \mathbb{E}_P \left[ \pi^{-1}GY^*\ell'_s(S^* \mid X; 0) \right], \tag{A.14}
$$

We proceed by verifying this condition for a choice of \( \phi(S, X) \) that satisfies (A.12), as required. Consider the function

$$
\phi(s, x) = \frac{\gamma(x)}{\pi}(\mu(s, x) - \mu(x)).
$$

Then \( \mathbb{E}_P[\phi(S^*, X) \mid X] = 0 \), which satisfies (A.12). Moreover,

$$
\mathbb{E}_P[\phi(S^*, X)\ell'_s(S^* \mid X; 0)] = \mathbb{E}_P \left[ \frac{\gamma(X)}{\pi}\mu(S^*, X)\ell'_s(S^* \mid X; 0) \right]
$$

$$
= \mathbb{E}_P \left[ \frac{\mathbb{E}_P[G \mid S^*, X]}{\pi} \mathbb{E}_P[Y^*\ell'_s(S^* \mid X; 0) \mid S^*, X] \right]
$$

$$
= \mathbb{E}_P \left[ \pi^{-1}GY^*\ell'_s(S^* \mid X; 0) \right],
$$

where the second to last equality follows by Assumption 2.2, which verifies (A.14) and thereby (A.6).

**Condition (A.7):** We derive the restrictions on \( \varphi(x) \) implied by (A.7). Note that the conjectured influence function (A.11) can be written as

$$
\psi(b_1) = gw \cdot \alpha(y, s, x) + (1 - g)w \cdot \beta(s, x) + g\varphi(x),
$$

where

$$
\mathbb{E}_P[\alpha(Y, S, X) \mid S = s, X = x, G = 1, W = 1] = \mathbb{E}_P[\beta(S, X) \mid G = 0, W = 1, X = x] = 0.
$$

We first show that the terms in (A.7) involving \( \alpha \) and \( \beta \) are zero. Indeed, for terms involving \( \alpha \), we have that

$$
\mathbb{E}_P[GW \cdot \alpha(Y, S, X)\ell'_s(G, X)] = \pi\mathbb{E}_P[\alpha(Y, S, X)\ell'_s(G, X) \mid G = 1, W = 1]P(W = 1 \mid G = 1) = 0
$$

since \( \mathbb{E}_P[\alpha(Y, S, X)\ell'_s(G, X) \mid G = 1, W = 1] = 0 \). Analogously, we have that

$$
\mathbb{E}[(1 - G)W \beta(S, X)\ell'_s(G, X; 0)] = 0.
$$

Therefore, (A.7) is equivalent to

$$
\frac{1}{\pi} \mathbb{E}_P[GY^*\ell'_s(G, X)] - \theta_{1,1}E[\ell'_s(G, X) \mid G = 1] = \mathbb{E}_P[GY^*\ell'_s(G, X)]. \tag{A.15}
$$

We proceed by verifying this condition for a choice of \( \varphi(X) \).

Consider the function

$$
\varphi(x) = \frac{1}{\pi}(\mu(x) - \theta_{1,1}),
$$

which satisfies (A.13). We can compute

$$
\mathbb{E}_P[\pi^{-1}\mu(X)\ell'_s(G, X)] = \mathbb{E}_P \left[ \frac{G}{\pi} \mathbb{E}_P[\mu(S^*, X) \mid G = 0, W = 1, X]\ell'_s(G, X; 0) \mid X \right]
$$

$$
= \mathbb{E}_P \left[ \frac{G}{\pi} \ell'_s(G, X; 0)\mathbb{E}_P[Y \mid W = 1, G = 1, S^*, X] \mid X \right]
$$
by iterated expectation conditional on Condition (A.9) and thereby (A.7).

**Condition (A.8):** Since the first and second terms \( \alpha(Y, S, X) \) and \( \beta(S, X) \) in the conjectured efficient influence function (A.11) are mean zero conditional on \( S^* \) and \( X \), respectively, the right-hand side of (A.8) is equal to

\[
\mathbb{E}_P \left[ \varphi(X) \left( GW \frac{q'(S, X)}{q(S, X)} - G(1 - W) \mathbb{E}_{P_x} \left[ q'(S^*, X) | X \right] \right) \right]
\]

\[
= \mathbb{E}_P \left[ \varphi(X) \gamma(X) \mathbb{E}_{P_x} \left[ q(S^*, X) \frac{q'(S, X)}{q(S^*, X)} - (1 - q(S^*, X)) \mathbb{E}_{P_x} \left[ q'(S^*, X) | X \right] \right] | G = 1, S^*, X \right) \]

\[
= \mathbb{E}_P \left[ \varphi(X) \gamma(X) \left( q(S^*, X) - (1 - q(S^*, X)) \mathbb{E}_{P_x} \left[ q'(S^*, X) | X \right] \right) \right] = 0
\]

by iterated expectation conditional on \( X \).

**Condition (A.9):** The only term on the right-hand side of (A.9) is

\[
\mathbb{E}_P[(1 - G)W \cdot \beta(S, X) \cdot (W - e(X))\delta(X)]
\]

for some function \( \delta(\cdot) \), where \((1 - G)W \cdot \beta(S, X)\) is the second term in the conjectured efficient influence function (A.11). We have:

\[
\mathbb{E}_P[(1 - G)W \cdot \beta(S, X) \cdot (W - e(X)) \cdot \delta(X)]
\]

\[
= \mathbb{E}_P[\mathbb{E}_P[W \cdot \beta(S, X) \cdot (W - e(X)) \cdot \delta(X) | G = 0, X](1 - \gamma(X))] \]

\[
= \mathbb{E}_P[\beta(S^*, X)e(X)(1 - e(X)) \cdot \delta(X)] = 0
\]

by the condition \( \mathbb{E}_{P_x} [\beta(S^*, X) | X] = 0 \), completing the proof.

### A.3. Proof of Theorem 3.2.

We maintain the notation from the proof of Theorem 3.1. Let \( \tilde{\psi}_{1,1}(b_1) \) be an influence function for \( \tilde{\theta}_{1,1} \). It is necessarily that case that

\[
\tilde{\psi}_{1,1}(b_1) = \psi_{1,1}(b_1) + f(b_1),
\]

where \( \psi(b_1) \) is the efficient influence function in derived in Theorem 3.1 and \( f(b_1) \) is mean-zero and orthogonal to the tangent space \( \mathcal{T} \), defined in the proof of Theorem 3.1. Moreover, we may write

\[
f(b_1) = gw f_1(y, s, x) + (1 - g)w f_2(s, x) + g(1 - w)f_3(x)
\]

\[
+ (1 - g)(1 - w)f_4(x) + A,
\]

(A.16)

where \( A \) is a constant making \( f \) mean-zero.

The orthogonality between \( f(b_1) \) and the tangent space \( \mathcal{T} \) implies the following set of conditions that are analogous to the conditions (A.5) through (A.9) defined in the proof of Theorem 3.1:

\[
0 = \mathbb{E}_P \left[ f(b_1)GW \ell'_* (Y | S, X) \right],
\]

(A.17)

\[
0 = \mathbb{E}_P \left[ f(b_1) \left( W \ell'_* (S | X) - G(1 - W) \frac{\int q(s, X)p'_*(s | X) d\nu_S(s)}{1 - \int q(s, X)p_*(s | X) d\nu_S(s)} \right) \right],
\]

(A.18)
In the sequel, we characterize \( f \) by deriving restrictions from each of these conditions for particular choices of the scores \( \ell_*(\cdot) \), \( e'(X) \), and \( q'(S,X) \).

Combining (A.17) and (A.16) yields

\[
0 = \mathbb{E}_P [GW f_1(Y, S, X) \ell'_* (Y \mid S, X)] = 0.
\]

As \( \ell'_*(Y \mid S, X) \) ranges over all conditionally mean-zero, square integrable functions, we may pick

\[
\ell'_*(Y \mid S, X) = f_1(Y, S, X) - \mathbb{E}_P[f_1(Y, S, X) \mid G = 1, W = 1, S, X],
\]

from which we find that

\[
0 = \mathbb{E}_P [GW f_1(Y, S, X) \ell'_* (Y \mid S, X)]
= \mathbb{E}_P [GW f_1^2(Y, S, X) - f_1(Y, S, X) \mathbb{E}_P[GW f_1(Y, S, X) \mid G = 1, W = 1, S, X]],
\]

and so \( f_1(Y, S, X) = f_1(S, X) \) does not depend on \( Y \).

Next, observe that plugging (A.16) into (A.20), yields

\[
0 = \mathbb{E}_P \left[ \gamma(X) \left(f_1(S^*, X) q'(S^*, X) - f_3(X) \mathbb{E}_P[q'(S^*, X) \mid X]\right) \right]
\]

after simplification via law of iterated expectations. Choosing

\[
q'(S^*, X) = \gamma(X)^{-1} \left(f_1(S^*, X) - \mathbb{E}_P[f_1(S^*, X) \mid X]\right)
\]

establishes that

\[
0 = \mathbb{E}_P [f_1^2(S^*, X) - f_1(S^*, X) \mathbb{E}_P[f_1(S^*, X) \mid X]],
\]

and so \( f_1(S, X) = f_1(X) \) does not depend on \( S \). This then implies that \( f_1(x) = f_3(x) \) by law of iterated expectations.

Now, plugging (A.16) into (A.18) yields

\[
0 = \mathbb{E}_P \left[ (1 - \gamma(X)) e(X) f_2(S^*, X) \ell'_* (S^* \mid X) \right],
\]

again after simplification via law of iterated expectations. As we may choose

\[
\ell'_*(S^* \mid X) = f_2(S^*, X) - \mathbb{E}_P[f_2(S^*, X) \mid X],
\]

we find similarly that \( f_2 \) is a function only of \( X \).

Plugging (A.16) into (A.21), gives the condition

\[
0 = \mathbb{E}_P \left[ (1 - \gamma(X))(f_2(X) - f_4(X)) e'(X) \right]
\]

where picking \( e'(X) = f_2(X) - f_4(X) \) implies that \( f_4(x) = f_2(x) \).

Lastly, plugging (A.16) into (A.19), we find

\[
0 = \mathbb{E}_P \left[ \ell'_*(G, X)(G f_3(X) + (1 - G)f_2(X)) \right].
\]

We then find that

\[
\mathbb{E}_P[\ell'_*(G, X)] = 0 = \mathbb{E}_P[\gamma(X)\ell'_*(1, X) + (1 - \gamma(X))\ell'_*(0, X)],
\]

and so \( f_1(x) = f_2(x) = f_3(x) = f_4(x) = C \) almost everywhere, for some constant \( C \). Hence, \( f(b) = 0 \) almost everywhere and the space of influence functions is a singleton.
A.4. Proof of Remark 3.4. We verify that the choice of Riesz representer
\[
\alpha(b) = \theta_{1,1} - \frac{g}{\pi} \left( \frac{w \mu_1(s, x)}{q_1(s, x)} - (\mu_1(x) - \theta_{1,1}) \right) + \frac{1 - g}{\pi} \left( \frac{\gamma(x)}{1 - \gamma(x)} \frac{w((\mu_1(s, x) - \mu_1(x))}{e(x)} \right)
\]  
\tag{A.22}
\]  

satisfies the necessary condition
\[
\mathbb{E}_P[\alpha(B) \ell'(B)] = M'(P, \theta_{1,1}, \hat{q}_1) = -\mathbb{E}_P \left[ \frac{\gamma(X) \mu(S^*, X) q'(S, X)}{q(S, X)} \right] - \theta_{1,1} \mathbb{E}_P[\ell'_*(G, X) \mid G = 1]
\]  

and is mean-zero. Observe that \(\mathbb{E}_P[\alpha(B)] = 0\) from an immediate calculation.

We compute
\[
\mathbb{E}_P[\alpha(B) \ell'(B)] = \mathbb{E}_P \left[ -\frac{GW}{\pi q(S, X)} \mu(S, X) \ell'_*(Y \mid S, X) \right] + \mathbb{E}_P \left[ \left( \frac{(1 - G)W \gamma(X)}{(1 - \gamma(X))e(X)\pi} (\mu(S, X) - \mu(X)) \right) \right.
\]
\[
- \frac{GW}{\pi q(S, X)} \mu(S, X) + \frac{GW}{\pi} (\mu(x) - \theta_{1,1}) \ell'_*(S \mid X) \right] \]
\[
+ \mathbb{E}_P \left[ G(1 - W) \mu(X) - \theta_{1,1} \mathbb{E}_P[q(S, X) \ell'_*(S \mid X) \mid X] \right]
\]
\[
- \mathbb{E}_P \left[ \frac{GW}{\pi q(S, X)} \mu(S, X) \left( \frac{q'(S, X)}{q(S, X)} \right) \right] + \mathbb{E}_P \left[ \frac{G}{\pi} (\mu(X) - \theta_{1,1}) \left( \frac{W q'(S, X)}{q(S, X)} - (1 - W) \frac{\mathbb{E}_P[q'(S, X) \mid X]}{\mathbb{E}_P[q(S, X) \mid X]} \right) \right] \]
\[
+ \mathbb{E}_P \left[ \frac{(1 - G) \gamma(X)}{1 - \gamma(X)} \frac{W(W - e(X))}{\pi e(X)(1 - e(X))} \ell'(X)(\mu(S, X) - \mu(X)) \right].
\]  

By the conditional mean-zero property of the terms in \(\alpha\) and the score, we can see immediately that the first, sixth, and seventh terms are zero. The second and the third terms simplify to
\[
\mathbb{E}_P[\pi^{-1} \gamma(X) q(S, X)(\mu(X) - \theta_{1,1}) \ell'_*(S \mid X; 0)]
\]
\[
- \mathbb{E}_P \left[ \frac{\gamma(X)}{\pi} (\mu(X) - \theta_{1,1}) \mathbb{E}_P[q \ell'_*(S \mid X; 0) \mid X] \right] = 0.
\]  

The fourth term simplifies to
\[
\mathbb{E}_P \left[ \frac{\gamma(X)}{\pi} (\mu(X) - \mu(S, X)) \ell'_*(1, X) \right] - \theta_{1,1} \mathbb{E}_P \left[ \frac{G}{\pi} \ell'_*(1, X) \right]
\]
\[
= -\theta_{1,1} \mathbb{E}_P \left[ \frac{G}{\pi} \ell'_*(1, X) \right]
\]
\[
= -\theta_{1,1} \mathbb{E} \left[ \ell'_*(1, X) \mid G = 1 \right].
\]  

and the fifth term simplifies to
\[
-\mathbb{E}_P \left[ \frac{\gamma(X) \mu(S^*, X) q'(S, X)}{\pi q(S, X)} \right],
\]  

which agrees with the two terms we desire, completing the proof. \(\blacksquare\)
The pathwise derivative of this parameter is given by Assumptions 2.1 to 2.3 and 2.5, the target parameter \( \theta \) is indexed by \( \eta \). With \( \gamma(x) \) as in Assumption 2.3, we have \( \ell(x) \), \( \gamma(x) \), \( g \), \( b \), and \( \mu \) as in Assumption 2.3 and Assumption 2.5 hold for each \( \eta \in \mathbb{R} \). Thus, the tangent space \( T_{\eta} \) is given by the mean-square closure of the linear span of the functions

\[
\ell(y) = g \ell(y \mid x, G = 1) + g \ell(s \mid x, G = 1) + w(1 - g) \ell(s \mid W = 1, x, G = 0) + \ell(x)
\]

where the functions \( s_1 \) through \( s_6 \) range over the space of mean-zero and square integrable functions that satisfy the restrictions

\[
0 = \mathbb{E}_P[s_1(Y \mid S, X, G = 1) \mid S, X, G = 1]
\]

\[
= \mathbb{E}_P[s_1(Y \mid S, X, G = 1) \mid S, X],
\]

\[
0 = \mathbb{E}_P[s_2(S \mid X, G = 1) \mid X, G = 1],
\]

\[
0 = \mathbb{E}_P[s_3(S \mid W = 1, X, G = 0) \mid W = 1, X, G = 0]
\]

\[
= \mathbb{E}_P[s_3(S^* \mid W = 1, X, G = 0) \mid X, G = 0],
\]

\[
0 = \mathbb{E}_P[s_4(X)],
\]

and \( s_5 \) and \( s_6 \) are unconstrained, where (A.26) and (A.28) follow from Assumption 2.3 and Assumption 2.1, respectively.

**Pathwise Differentiability Representation.** Recall from the proof of Theorem 2.1 that, by Assumptions 2.1 to 2.3 and 2.5, the target parameter \( \theta_{1,1} \) can be written as

\[
\theta_{1,1} = \mathbb{E}_{P_{\eta}}[Y^* \mid G = 1]
\]

\[
= \mathbb{E}_P[\mathbb{E}_P[Y \mid S, X, G = 1] \mid W = 1, X, G = 0] \mid G = 1
\]

\[
= \iint g(y \mid s, x, G = 1)p(s \mid x, W = 1, G = 0)p(x \mid G = 1) dv_Y(y) dv_S(s) dv_X(x).
\]

The pathwise derivative of this parameter is given by

\[
\theta'_{1,1} = \mathbb{E}_P[\mathbb{E}_P[Y \mid U(Y, S, X) \mid S, X, G = 1] \mid X, W = 1, G = 0] \mid G = 1
\]
with
\[
U(y, s, x) = \ell'(y \mid s, x, G = 1) + \ell'(s \mid x, W = 1, G = 0) + \ell'(x \mid G = 1)
\]
\[
= \ell'(y \mid s, x, G = 1) + \ell'(s \mid x, W = 1, G = 0) + \ell'(x) + \frac{\gamma'(x)}{\gamma(x)} - \frac{E_P[\gamma'(X) + \gamma(X)\ell'(X)]}{\pi},
\]
where final three terms follow from
\[
p'(x \mid G = 1) = \ell'(x \mid G = 1)p(x \mid G = 1)
\]
and Bayes’ rule in the form of
\[
\ell(x \mid G = 1) = \log \gamma(x) + \ell(x) - \log \int \gamma(x)p(x) \, d\nu_X(x).
\]
Observe that, using the definitions of \(\mu(s, x)\) and \(\mu(x)\), we may further simplify the pathwise derivative to give
\[
\theta'_{1,1} = E_P[E_P[E_P[Y \ell'(Y \mid S, X, G = 1) \mid S, X, G = 1] \mid X, W = 1, G = 0] \mid G = 1]
\]
\[
\quad + E_P[E_P[E_P[\mu(S, X)\ell'(S \mid X, G = 0, W = 1) \mid X, W = 1, G = 0] \mid G = 1]
\]
\[
\quad + \pi^{-1}E_P[\mu(X) (\gamma'(X) + \gamma(X)\ell'(X))] - \pi^{-1}E_P[\gamma'(X) + \gamma(X)\ell'(X)]\theta_{1,1}. \tag{A.30}
\]

An influence function for \(\theta_{1,1}\) is a mean-zero and square-integrable function \(\xi_{1,1}(B_1)\) that satisfies the condition
\[
\theta'_{1,1} = E[\xi(B_1)\ell'(B_1)]. \tag{A.31}
\]
We make a conjecture for such a function and verify that it satisfies this condition and is an element of the tangent space \(T\), establishing that it is the efficient influence function.

**Conjectured Efficient Influence Function.** We conjecture that the efficient influence function takes the form
\[
\xi_{1,1}(b_1) = g(y - \mu(s, x)) \cdot f_1(s, x) + w(1 - g)(\mu(s, x) - \mu(x)) \cdot f_2(x) + g \cdot f_3(x), \tag{A.32}
\]
for functions \(f_1(s, x), f_2(x),\) and \(f_3(x)\) to be specified, where \(f_3\) will be chosen to satisfy
\[
E_P[f_3(X) \mid G = 1] = 0 = E_P[f_3(X)\gamma(X)]. \tag{A.33}
\]
Observe that for this choice, \(\xi_{1,1}(b)\) will be an element of the tangent space \(T\). This follows as
\[
s_1(y \mid s, x, G = 1) = (y - \mu(s, x))f_1(s, x)
\]
satisfies (A.26), we make the choice \(s_2(s \mid X, G = 1) = 0\) satisfying (A.27),
\[
s_3(s \mid W = 1, x, G = 0) = (\mu(s, x) - \mu(x))f_2(x)
\]
satisfies (A.28), and that setting \(s_4(x) = \gamma(x)f_3(x)\) and \(s_5(x) = \gamma(x)(1 - \gamma(x))f_3(x)\) yields
\[
gf_3(x) = s_4(x) + \frac{g - \gamma(x)}{\gamma(x)(1 - \gamma(x))} s_5(x),
\]
satisfying (A.29), the mean zero conditions for \(s_4\) and \(s_5\) by (A.33), and implicitly satisfying (A.27) and the mean zero condition for \(s_0\).

**Verification of Pathwise Differentiability Representation.** In the sequel, we make choices of \(f_1(s, x), f_2(x),\) and \(f_3(x)\) such that the pathwise derivative of \(\theta_{1,1}\) satisfies (A.31) and (A.33), completing the proof.

Our conjecture for \(\xi_{1,1}(b_1)\) allows for some simplification of the right-hand side of (A.31). Note that, for any measurable \(h(S, X)\) and \(k(X)\), we have
\[
E_P[GF(Y - \mu(S, X))f_1(S, X)h(S, X)] = 0 = E_P[(1 - G)W(\mu(S, X) - \mu(x))f_2(X)k(X)].
\]
by iterated expectations. Thus, by additionally applying the mean-zero property of the scores \( \ell'(Y \mid S, X, G = 1) \), \( \ell'(S \mid X, G = 1) \), we can simplify the pathwise differentiability condition to give

\[
\begin{align*}
\mathbb{E}_P [\xi_{1,1}(B_1) \ell'(B_1)] &= \mathbb{E}_P [G(Y - \mu(S, X)) f_1(S, X) \ell'(Y \mid S, X, G = 1)] \\
&+ \mathbb{E}_P [W(1 - G)(\mu(S, X) - \mu(X)) f_2(X) \ell'(S \mid W = 1, X, G = 0)] \\
&+ \mathbb{E}_P [(G\ell'(X) + \gamma'(X)) f_3(X)].
\end{align*}
\]  
(A.34)

Therefore, we obtain a set of sufficient conditions for (A.31) by matching each term in (A.34) with (A.30), giving

\[
\begin{align*}
\mathbb{E}_P [G(Y - \mu(S, X)) f_1(S, X) \ell'(Y \mid S, X, G = 1)] &= \mathbb{E}_P [\mathbb{E}_P[Y \ell'(Y \mid S, X, G = 1) \mid S, X, G = 1] \mid X, W = 1, G = 0] \mid G = 1 \\
\mathbb{E}_P [W(1 - G)(\mu(S, X) - \mu(X)) f_2(X) \ell'(S \mid W = 1, G = 0, X)] &= \mathbb{E}_P [\mathbb{E}_P[\mu(S, X) \ell'(S \mid X, W = 1, G = 0) \mid X, W = 1, G = 0] \mid G = 1] \\
\mathbb{E}_P [(G\ell'(X) + \gamma'(X)) f_3(X)] &= \mathbb{E}_P \left[ \pi^{-1}\mu(X) (\gamma'(X) + \ell'(X)\gamma(X)) \right] - \pi^{-1}\mathbb{E}_P[\gamma'(X) + \gamma(X)\ell'(X)]|\theta_{1,1}.
\end{align*}
\]  
(A.37)

We complete the proof by considering each condition sequentially, making appropriate choices of \( f_1(s, x), f_2(x), \) and \( f_3(x) \), where we note that each function appears in only one term.

**Condition (A.35):** Evaluating the left-hand side of (A.35) we find that

\[
\begin{align*}
\mathbb{E}_P [G(Y - \mu(S, X)) f_1(S, X) \ell'(Y \mid S, X, G = 1)] &= \mathbb{E}_P [GY f_1(S, X) \ell'(Y \mid S, X, G = 1)] \\
&= \mathbb{E}_P[\gamma(X) \mathbb{E}_P[f_1(S, X) \mathbb{E}_P[Y \ell'(Y \mid S, X, G = 1) \mid S, X, G = 1] \mid S, X, G = 1]].
\end{align*}
\]

Thus, if we choose

\[
f_1(s, x) = \frac{p(s \mid x, W = 1, G = 0)}{\gamma(x)p(s \mid x, G = 1)} \frac{p(x \mid G = 1)}{p(x)}
\]

to change the measure of the two outer expectations, (A.35) is satisfied. Bayes’ rule calculations show that

\[
f_1(s, x) = \frac{1}{\pi} \frac{r(s, x)}{e(x)} \frac{t(s, x)}{1 - t(s, x)} \frac{\gamma(x)}{1 - \gamma(x)},
\]

where we recall that \( r(s, x) = P(W = 1 \mid S = s, X = x, G = 0) \) is the surrogate score, and \( t(s, x) = P(G = 0 \mid S = s, X = x) \) is the sampling score.

**Condition (A.36):** Observe that, for any measurable function \( h(s, x) \), we have that

\[
\mathbb{E}_P[W(1 - G)h(S, X)] = \mathbb{E}_P[e(X)(1 - \gamma(X))\mathbb{E}_P[h(S, X) \mid X, W = 1, G = 0]].
\]

Also, recall that since \( \ell'(S \mid X, W = 1, G = 0) \) is a conditional score, it is conditionally uncorrelated with any function of \( X \); that is, for any measurable \( h(x) \),

\[
\mathbb{E}_P[h(x)\ell'(S \mid X, W = 1, G = 0) \mid X, W = 1, G = 0] = 0
\]

almost surely.

With the above two observations in mind, evaluating the left-hand side of (A.36), we find that

\[
\begin{align*}
\mathbb{E}_P [W(1 - G)(\mu(S, X) - \mu(X)) f_2(X) \ell'(S \mid W = 1, G = 0, X)] &= \mathbb{E}_P [e(X)(1 - \gamma(X))\mathbb{E}_P[\mu(S, X) - \mu(X)) f_2(X) \ell'(S \mid W = 1, G = 0, X)] \\
&= \mathbb{E}_P [e(X)(1 - \gamma(X)) f_2(X) \mathbb{E}_P[\mu(S, X) \ell'(S \mid X, W = 1, G = 0) \mid X, W = 1, G = 0]]
\end{align*}
\]

Thus, again, if we choose

\[
f_2(x) = \frac{p(x \mid G = 1)}{p(x)e(x)(1 - \gamma(x))} = \frac{1}{\pi} \frac{1}{e(x)} \frac{\gamma(x)}{1 - \gamma(x)},
\]
again to change the measure of the outer expectation, then \( A.36 \) is satisfied.

**Condition (A.37):** Lastly, consider the choice
\[
\xi_{1,1}(b) = \frac{g}{\pi} \left( \frac{\gamma(x)}{1 - \gamma(x)} \frac{r(s,x) t(s,x)}{1 - t(s,x)} \frac{(y - \mu(s,x))}{e(x)} + (\mu(x) - \theta_{1,1}) \right) + \frac{1 - g}{\pi} \left( \frac{w(\mu(s,x) - \mu_1(x))}{e(x)} \right),
\]
as desired.

**A.6. Proof of Theorem 3.4.** We demonstrate that \( \xi_{1,1} \) is the efficient influence function for \( \theta_{1,1} \). An analogous argument verifies the statement for \( \theta_{0,1} \), and therefore for \( \tau_1 \). We maintain the notation from the proof of Theorem 3.3.

Let \( \xi_1(b_1) \) be an influence function for \( \theta_{1,1} \). It is necessarily the case that
\[
\xi_{1,1}(b_1) = \xi_{1,1}(b_1) + f(b_1),
\]
where \( \xi_{1,1}(b_1) \) is the efficient influence function derived in Theorem 3.3, and \( f(b_1) \) is mean-zero and orthogonal to the tangent space \( T \), defined in the proof of Theorem 3.3. We may without loss of generality write
\[
f(b) = g \cdot f_1(y, s, x) + (1 - g) w \cdot f_2(s, x) + (1 - g)(1 - w) \cdot f_3(x) + A,
\]
where \( A \) is some constant making \( f \) mean-zero.

The orthogonality between \( f \) and the tangent space \( T \) implies that
\[
\mathbb{E}_P[f(B_1)] \ell'(B_1) = 0
\]
for any \( \ell'(\cdot) \) corresponding to a score of a parametric submodel.

We first argue that \( f_1(y, s, x) = f_1(x) \) does not depend on \( y \) or \( s \). Consider all free components in the score \( A.24 \), and set everything except for \( \ell'(y | s, x, G = 1) \) to zero. Then the orthogonality condition \( A.38 \) becomes
\[
\mathbb{E}_P[G f_1(Y, S, X) \ell'(Y | S, X, G = 1)] = 0.
\]
Picking \( \ell'(Y | S, X, G = 1) = f_1(Y, S, X) - \mathbb{E}_P[f_1(Y) | S, X, G = 1] \) implies that \( \mathbb{E}_P[G f_1(Y, S, X) \ell'(Y | S, X, G = 1]|S, X, G = 1] > 0 \) unless \( f_1(Y, S, X) = \mathbb{E}[f_1(Y) | S, X, G = 1] \) almost surely. Therefore \( f_1(Y, S, X) \) does not depend on \( Y \). Similarly, if we set every free component in \( A.24 \) except for \( \ell'(S | X, G = 1) \) to zero, then choosing \( \ell'(S | X, G = 1) = f_1(S, X) - \mathbb{E}[f(S, X) | X, G = 1] \) again implies that \( f_1 \) only depends on \( X \). A similar argument with the score component \( \ell'(S | W = 1, X, G = 0) \) shows that \( f_2(s, x) = f_2(x) \) depends only on \( x \). Therefore, \( f(b) \) must take the form
\[
f(b) = g \cdot f_1(x) + (1 - g) w \cdot f_2(x) + (1 - g)(1 - w) \cdot f_3(x).
\]

The orthogonality condition \( A.38 \) then takes the form
\[
0 = \mathbb{E}_P[f(B) \ell'(X)] + \mathbb{E}_P\left[f(B) \frac{G - \gamma(X)}{\gamma(X)(1 - \gamma(X))} \gamma'(X)\right] \\
+ \mathbb{E}_P\left[(1 - G)(W f_2(X) + (1 - W)f_3(x)) \frac{W - e(X)}{e(X)(1 - e(X))} e'(X)\right].
\]
for any valid choices of \( \ell'(x), \gamma'(x), e'(x) \), as the other terms are zero by the mean-zero conditions on the conditional scores.

Observe that the third term can be simplified as

\[
\mathbb{E}_P \left[ \frac{(1 - G)(W f_2(X) + (1 - W)f_3(X))}{e(X)(1 - e(X))} \right] - e(X) \gamma'(X)
\]

\[
= \mathbb{E}_P \left( f_2(X) - f_3(X) \right) - e(X) \gamma'(X) | G = 0.
\]

Picking \( e'(x) = f_2(x) - f_3(x) \) implies that \( f_2(x) = f_3(x) \). Thus \( f(b) = g f_1(x) + (1 - g)f_2(x) \). Working with the second term in (A.39) (i.e. setting \( \ell'(x) = 0 = e'(x) \) and considering choices of \( \gamma'(x) \)) yields that

\[
\mathbb{E}_P \left( f(B) \frac{G - \gamma(X)}{\gamma(X)(1 - \gamma(X))} \gamma'(X) \right) = \mathbb{E}_P [(f_1(X) - f_2(X)) \gamma'(X)]
\]

Picking \( \gamma'(x) = f_1(x) - f_2(x) \) shows that \( f_1(x) = f_2(x) \). Therefore, from the first term in (A.39), we find that for all \( \ell' \) s.t. \( \mathbb{E}_P[\ell'(X)] = 0 \),

\[
\mathbb{E}_P[f(B)\ell'(X)] = \mathbb{E}_P[f_1(X)\ell'(X)] = 0 \implies f_1(X) = 0.
\]

implies that \( f_1(X) = 0 \) almost everywhere. Hence, \( f(b_1) = 0 \) almost everywhere, completing the proof.

**APPENDIX B: LONG-TERM AVERAGE TREATMENT EFFECT IN THE EXPERIMENTAL SAMPLE**

In this section, we develop results analogous to those presented in the main text for long-term treatment effect for units in the observational sample, given by

\[
\tau_0 = \mathbb{E}_{P_\pi} [Y_i(1) - Y_i(0) | G_i = 0].
\]

This estimand was considered in Athey et al. (2020b) for the case that treatment is not observed in the observational sample. The efficient influence functions and efficiency bounds we state in this section for that context correct those given in the Theorem 1 and Theorem 3 of the February, 2020 draft of that paper.

**B.1. Identification.** The identifying assumptions for \( \tau_0 \) are slightly less restrictive than for \( \tau_1 \). In particular, if treatment is not observed in the observational sample, it is unnecessary to impose the restriction Assumption 2.2.

**Theorem B.1.** The estimand of interest \( \tau_0 \), defined in Equation (2.1), is identified under the following conditions:

1. (Athey et al., 2020a) Under Assumptions 2.1, 2.2, and 2.4 and if treatment is observed in the observational sample, then \( \tau_0 \) is point identified.
2. (Athey et al., 2020b) Under Assumptions 2.1, 2.3, and 2.5 and if treatment is not observed in the observational sample, then \( \tau_0 \) is point identified.

**Proof.** The result follows almost immediately from inspection of the proof of Theorem 2.1 by considering the parameter

\[
\theta_{0,1} = \mathbb{E}_{P_\pi} [Y(1) | G = 0].
\]

In the case that treatment is observed in the observational sample, the only difference will occur in the final step where it is equally as apparent that

\[
\mathbb{E}_{P_\pi} [\mu_1(X) | G = 0]
\]

is identified. In the case that treatment is not observed in the observational sample the only difference is that Assumption 2.2 need not be invoked in order to condition on \( G = 0 \), where again the expectation of \( \mathbb{E}_P[\mu(X) | X, W = 1, G = 0] \) conditional on \( G = 0 \) remains identified.
B.2. Semiparametric Efficiency. We state two theorems that are analogous to Theorems 3.1 and 3.3 for the case where the estimand of interest is the average treatment effect conditional on being in the experimental sample: \( \tau_0 = \theta_{1,0} - \theta_{0,0} = \mathbb{E}[Y(1) - Y(0) \mid G = 0] \).

First, we treat the case that treatment is observed in the observational sample, providing an analogue of Theorem 3.1.

Theorem B.2. Under Assumptions 2.1, 2.2, and 2.4 and if treatment is observed in the observational sample, the efficient influence function for the parameter \( \theta_{1,0} \) is
\[
\psi_{1,0}(b) = \frac{g}{1 - \pi} \left( 1 - \gamma(x) \right) \frac{w(y - \mu_1(s, x))}{q_1(s, x)} \\
+ \frac{1 - g}{1 - \pi} \left( \frac{w(\mu_1(s, x) - \mu_1(x))}{e(x)} + (\mu_1(x) - \theta_{1,0}) \right) .
\] (B.1)

Corollary B.1. Under Assumptions 2.1, 2.2, and 2.4 and if treatment is observed in the observational sample, the efficient influence function for the parameter \( \tau_0 \) is
\[
\psi_0(b) = \frac{g}{1 - \pi} \left( 1 - \gamma(x) \right) \left( \frac{w(y - \mu_1(s, x))}{q_1(s, x)} - \frac{(1 - w)(\mu_0(s, x) - \mu_0(x))}{q_0(s, x)} \right) \\
+ \frac{1 - g}{1 - \pi} \left( \frac{w(\mu_1(s, x) - \mu_1(x))}{e(x)} - \frac{(1 - w)(\mu_0(s, x) - \mu_0(x))}{1 - e(x)} + (\mu_1(x) - \mu_1(x) - \tau_0) \right) .
\] (B.2)

Next, we treat the case that treatment is not observed in the observational sample, providing an analogue of Theorem 3.3.

Theorem B.3. Under Assumptions 2.1, 2.3, and 2.5 and if treatment is not observed in the observational sample, the efficient influence function for the parameter \( \theta_{1,0} \) is
\[
\xi_{1,0}(b) = \frac{g}{1 - \pi} \left( \frac{r(s, x) t(s, x)}{1 - t(s, x)} \right) \left( \frac{y - \mu(s, x)}{e(x)} \right) \\
+ \frac{1 - g}{1 - \pi} \left( \frac{w(\mu(s, x) - \mu_1(x))}{e(x)} + (\mu_1(x) - \theta_{1,0}) \right) .
\] (B.3)

Corollary B.2. Under Assumptions 2.1, 2.3, and 2.5 and if treatment is not observed in the observational sample, the efficient influence function for the parameter \( \tau_0 \) is
\[
\xi_0(b) = \frac{g}{1 - \pi} \left( \frac{t(s, x)}{1 - t(s, x)} \right) \left( \frac{r(s, x) - e(x)}{e(x)} \right) \left( \frac{y - \mu(s, x)}{1 - e(x)} \right) \\
+ \frac{1 - g}{1 - \pi} \left( \frac{w(\mu(s, x) - \mu_1(x))}{e(x)} + \frac{(1 - w)(\mu(s, x) - \mu_0(x))}{1 - e(x)} + (\mu_1(x) - \mu_0(x) - \tau_0) \right) .
\] (B.4)

In the following two sections, we prove Theorems B.2 and B.3 in a somewhat constructive manner, where restrictions sufficient for the efficient influence function are derived and the exact forms of the functions are guesses that we verify. One could arrive at the same conclusion by plugging in explicit expressions for \( \psi_{1,0} \) and \( \xi_{1,0} \) and verifying the conditions, but doing so obscures the deductive exercise in the efficient influence function calculations that may reveal certain statistical structures in the underlying problems. We maintain the same notation and closely follow the same structure as the proof of Theorems 3.1 and 3.3, emphasizing differences without repeating shared steps in the argument.

B.3. Proof of Theorem B.2. The analogue of (A.3) for \( \theta_{1,0} \) is
\[
\theta'_{1,0} = \mathbb{E}_{P_x}[Y^* t'(Y^* \mid S^*, X) \mid G = 0] \\
+ \mathbb{E}_{P_x}[Y^* t'(S^* \mid X) \mid G = 0] + \mathbb{E}_{P_x}[Y^* t'(X \mid G = 0) \mid G = 0] ,
\] (B.5)
with each term now conditioned on $G = 0$ and $\ell'(X \mid G = 0) \text{ replaces } \ell'(X \mid G = 1)$. Note also that
\[
\ell'_s(x \mid G = 0) = \ell'_s(x, G = 0) - \mathbb{E}_{P_s}[\ell'_s(X, G = 0) \mid G = 0].
\]

For a candidate influence function $\tilde{\psi}_{1,0}(b_1)$, the analogues of the pathwise differentiability conditions (A.5) through (A.9) are
\[
\frac{1}{1 - \pi} \mathbb{E}_{P_s}[(1 - G)Y^*\ell'_s(Y^* \mid S^*, X)] = \mathbb{E}_P[\tilde{\psi}_{1,0}(B_1)GW\ell'_s(Y \mid S, X)] \tag{B.7}
\]
\[
\frac{1}{1 - \pi} \mathbb{E}_{P_s}[(1 - G)Y^*\ell'_s(S^* \mid X)] = \mathbb{E}_P[\tilde{\psi}_{1,0}(B_1)\left(W\ell'_s(S \mid X) - G(1 - W)\frac{\int q(s, X)p'_s(s \mid X) d\nu(s)}{\int q(s, X)p_s(s \mid X) d\nu(s)}\right)] \tag{B.8}
\]
\[
\mathbb{E}_P[\tilde{\psi}_{1,0}(B_1)\ell'_s(G, X)] = \frac{1}{1 - \pi} \mathbb{E}_{P_s}[(1 - G)Y^*\ell'_s(G = 0, X)] - \theta_{1,0} \mathbb{E}_{P_s}[\ell'_s(G = 0, X) \mid G = 0] \tag{B.9}
\]
\[
0 = \mathbb{E}_P\left[\tilde{\psi}_{1,0}(B_1)G\left(W\frac{q'(S, X)}{q(S, X)} - (1 - W)\frac{\int q'(S, X)p'_s(s \mid X) d\nu(s)}{\int q(s, X)p_s(s \mid X) d\nu(s)}\right)\right] \tag{B.10}
\]
\[
0 = \mathbb{E}_P\left[\tilde{\psi}_{1,0}(B_1)(1 - G)\frac{W - e(X)}{e(X)(1 - e(X))} \ell'(X)\right] \tag{B.11}
\]

where the only differences relative to (A.5) through (A.9) are the replacement of $G$ with $1 - G$ and $\pi$ with $1 - \pi$ in (B.7) through (B.9) on the sides of the equalities that correspond to $\theta'_{1,0}$. Since the density of the data doesn’t change with the estimand, the sides of the equalities that correspond to orthogonality condition
\[
\mathbb{E}_P[\tilde{\psi}_{1,0}(B_1)\ell'(B_1)]
\]
remains unchanged relative to (A.5) through (A.9).

We make the same choices $s_4(s, x) = -q(s, x)s_2(s, x)$ and $s_5(x) = 0$ as in (A.10), resulting in an conjectured efficient influence function of the form
\[
\psi_{1,0}(b_1) = gw \cdot s_1(y, s, x) + (1 - g)w \cdot s_2(s, x) + s_3(g, x)
\]

We make the conjecture that
\[
s_1(y, s, x) = f_1(x) : \frac{y - \mu(s, x)}{q(s, x)}, \quad s_2(s, x) = \frac{\mu(s, x) - \mu(x)}{(1 - \pi)e(x)}, \quad \text{and}
\]
\[
s_3(g, x) = \frac{1 - g}{1 - \pi}(\mu(x) - \theta_{1,0}), \tag{B.12}
\]
where
\[
f_1(x) = \frac{1}{1 - \pi} \frac{1 - \gamma(x)}{\gamma(x)}.
\]

These choices satisfy the conditional mean-zero conditions for scores.

We verify the conditions (B.7) through (B.11) sequentially, completing the proof. To verify condition (B.7), we note that by the mean-zero property of the conditional scores, the right-hand side of (B.7) simplifies to
\[
\mathbb{E}_{P_s}\left[f_1(X)GWY^*\ell'_s(Y^* \mid S^*, X)\right] = \mathbb{E}_{P_s}[\pi f_1(X)Y^*\ell'_s(Y^* \mid S^*, X) \mid G = 1]
\]
right-hand side of (B.8) simplifies to
\[ \frac{1 - \gamma}{\gamma - 1 - \pi} Y^* \ell'(Y^* | S^*, X) \mid G = 1 \]
\[ = E_{P_\gamma} \left[ \frac{1 - \gamma}{\gamma - 1 - \pi} Y^* \ell'(Y^* | S^*, X) \mid G = 0 \right] \]
\[ = E \left[ Y^* \ell'(Y^* | S^*, X) \mid G = 0 \right] \]
\[ = \frac{1}{1 - \pi} E_{P_\gamma} \left[ (1 - G) Y^* \ell'_s(Y^* | S^*, X) \right] \]
\[ = E_P \left[ \psi_{1,0}(b_1) GW \ell'_s(Y | S, X) \right]. \]

where the third equality follows from the importance sampling argument.

\[ p_\gamma(y, s, x \mid G = 1) = p_\gamma(y, s \mid x)p_\gamma(x \mid G = 1) = p_\gamma(y, s \mid x)p_\gamma(x \mid G = 0) \frac{1 - \gamma}{\gamma - 1 - \pi}. \]

To verify condition (B.8), we note that by the mean-zero property of the conditional scores, the right-hand side of (B.8) simplifies to
\[ E_{P_\gamma} \left[ \frac{1 - G}{1 - \pi} W \mu(S, X) \ell'(S^* | X) \right] = E_{P_\gamma} [\mu(S, X) \ell'(S^* | X) \mid G = 0] \]
\[ = E_{P_\gamma} [Y^* \ell'(S^* | X) \mid G = 0], \]
which is equal to the left-hand side of (B.8).

To verify condition (B.9), we note again that by the mean-zero properties of the conditional scores, the left-hand side of (B.9) simplifies to
\[ E_P \left[ \frac{1 - G}{1 - \pi} (\mu(X) - \theta_{1,0}) \ell'(G = 0, X) \right] = E_P \left[ (\mu(X) - \theta_{1,0}) \ell'_s(G = 0, X) \mid G = 0 \right] \]
\[ = E_P [\mu(X) \ell'_s(G = 0, X) \mid G = 0] \]
\[ - \theta_{1,0} E_{P_\gamma} [\ell'_s(G = 0, X) \mid G = 0] \]
which is equal to the right-hand side of (B.9).

Finally, condition (B.10) holds by mean-zero properties of \( s_1(y, s, x) \) and condition (B.11) holds by mean-zero properties of \( s_2(y, s, x) \) and the fact that \( E_P[W - e(X) \mid X, G = 0] = 0 \), completing the proof.

B.4. Proof of Theorem B.3. The analogue of the pathwise derivative (A.30) for \( \theta_{1,0} \) is now
\[ \theta'_{1,0} = E_P [E_P [E_P[Y \ell'(Y \mid S, X, G = 1) \mid S, X, G = 1] \mid X, W = 1, G = 0] \mid G = 0] \]
\[ + E_P [E_P [\mu(S, X) \ell'(S \mid X, G = 0, W = 1) \mid X, W = 1, G = 0] \mid G = 0] \]
\[ + E_P \left[ (1 - \pi)^{-1} \mu(X) \left( -\gamma'(X) + \ell'(X)(1 - \gamma'(X)) \right) \right] \]
\[ + (1 - \pi)^{-1} E_P [\gamma'(X) + \gamma(X) \ell'(X)] \theta_{1,0}. \] (B.13)

The conjectured efficient influence function is given by
\[ \xi_{1,0}(b_1) = g(y - \mu(s, x)) \cdot f_1(s, x) + (1 - g)w(\mu(s, x) - \mu(x)) \cdot f_2(x) + (1 - g) \cdot f_3(x) \]
with \( f_1(s, x), f_2(x), \) and \( f_3(x) \) to be specified, where \( f_3(x) \) is chosen such that
\[ E[f_3(X) \mid G = 0] = 0. \]

An argument very similar to that given in Appendix A.5 demonstrates that \( \xi_{1,0}(b_1) \) is in the tangent space.

Again, following a similar argument, we may verify that the choices
\[ f_1(s, x) = \frac{r(s, x)}{(1 - \pi)e(x) \left( 1 - t(s, x) \right)}, \]
\[ f_2(x) = \frac{1}{(1 - \pi)e(x)}, \]

where the planner's choice of \( r(s, x) \) is in the tangent space.

where the third equality follows from the importance sampling argument.

\[ p_\gamma(y, s, x \mid G = 1) = p_\gamma(y, s \mid x)p_\gamma(x \mid G = 1) = p_\gamma(y, s \mid x)p_\gamma(x \mid G = 0) \frac{1 - \gamma}{\gamma - 1 - \pi}. \]
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\[ f_3(x) = \frac{\mu(x) - \theta_{1,0}}{1 - \pi}, \text{and} \]

satisfy the pathwise differentiability conditions. The conditions for \( f_1(s,x) \) and \( f_2(x) \) are such that multiplying the density ratio

\[
\frac{p(x \mid G = 0)}{p(x \mid G = 1)} = \frac{\pi}{1 - \pi} \frac{1 - \gamma(x)}{\gamma(x)}
\]

to the choices for \( f_1(s,x) \) and \( f_2(x) \) given in Appendix A.5 yields the corresponding choices here. ■